Maurolycus in Book I of his arithmetic begins with the definitions of different kinds of numbers, namely, *even, odd, triangular, square, numeri parte altera longiores*, etc. By definition the nth triangular number is the sum of the integers from 1 to n inclusive and the nth *numerus parte altera longior* is $n(n-1)$. He arranges them in a table as follows:

Integers	*Even*	*Odd*	*Triangular*	*Square*	*N. P. A. L.*[1]
1	0	1	1	1	0
2	2	3	3	4	2
3	4	5	6	9	6
4	6	7	10	16	12
5	8	9	15	25	20
6	10	11	21	36	30
7	12	13	28	49	42
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	E	O	T	S	L

I have added at the bottom of each column a symbol for the nth number of that column which I find convenient in explaining the work of Maurolycus. Numbers in the same row Maurolycus calls collateral numbers. A number in one row is said by him to *precede* any number in the following row and to *follow* any number in the preceding row; *e. g.*, he calls 15 the triangular number following the even number 6.

I quote a number of Maurolycus's theorems for reference. In some cases I give the proofs as given by him. The numbering of the theorems is his.

Proposition IV. "The odd numbers are obtained from unity by successive additions of 2." (Maurolycus uses this in *Proposition VI* in the form $O_n + 2 = O_{n+1}$, *i. e.*, the nth odd number plus 2 equals the next odd number.)

Proposition VI. "Every integer plus the preceding integer equals the collateral odd number." [In symbols this is $n + (n-1) = O_n$.] Maurolycus's proof, freely translated, is this:

"The integer 2 added to unity makes the integer 3 but when added to 3 it makes an amount greater by 2 and this (by virtue of *Proposition IV*) is the next odd integer, namely 5. Again since the integer 3 added to 2 makes 5, which is the collateral odd integer, when it is added to 4 the result will be greater by 2, that is (by virtue of *Proposition IV*), it will be the next odd integer which is 7. And in like manner to infinity as the proposition states."

This is not a very clear statement of a proof by mathematical induction but the idea is there. Maurolycus's ideas might be put more clearly as follows: The theorem is true by inspection in the case of the first two integers 1 and 2, *i. e.*, $2 + 1 = 3$ which is the odd integer collateral to 2. This is the first part of the induction proof. Maurolycus then takes up the special cases $3 + 2 = 5$ and $4 + 3 = 7$ and in doing so he shows by his repeated use of *Proposition IV* that

[1] *Numeri Parte Altera Longiores.*

the other part of the mathematical induction proof was in his mind. His *Proposition IV* furnishes the argument from n to $n + 1$. In modern notation it would be put in this way:

If $n + (n - 1) = O_n$ (*i. e.*, if any integer plus the preceding one equals the collateral odd integer), the result of adding $1 + 1$ to the left side and 2 to the right side is $(n + 1) + n = O_n + 2$. But by Prop. IV, $O_n + 2 = O_{n+1}$. Therefore $(n + 1) + n = O_{n+1}$.

This argument from n to $n + 1$ seems to have been in Maurolycus's mind. But if this were the only example of complete induction in his work it might not be a conclusive proof that he understood the method. *Proposition XV* is a much more convincing case. But before giving an account of it I wish to state several other of his theorems for reference and to discuss the proposition which Cantor says Pascal got from Maurolycus.

Proposition VIII. "Every triangular number doubled equals the following *numerus parte altera longior*." (In symbols this is $2T_n = L_{n+1}$.)

Proposition X. "Every *numerus parte altera longior* plus its collateral integer equals the collateral square number." (In symbols, $L_n + n = S_n$.)

Proposition XI. "Every triangular number plus the preceding triangular number equals the collateral square number." (In symbols, $T_n + T_{n-1} = S_n$.)

This proposition, although stated somewhat differently by Cantor, is the one which Cantor says Pascal got from Maurolycus and which he says Maurolycus proved by complete induction. For since a triangular number is equal to the sum of the natural numbers in order, $T_{n-1} = T_n - n$, and it follows that $T_n + T_{n-1} = 2T_n - n$; or, since by the formula for the sum of an arithmetic progression[1] the nth triangular number is $[n(n + 1)]/2$, the equation $T_n + T_{n-1} = S_n$ is equivalent to

$$2\left[\frac{n(n+1)}{2}\right] - n = S_n = n^2,$$

which is the form that Cantor gives. But Cantor is wrong in saying that this theorem was proved by Maurolycus by complete induction. For Maurolycus's proof (in modern notation) is this:

By definition $T_n = T_{n-1} + n$. Therefore $T_n + T_{n-1}$, the left-hand member of the relation to be proved, is equal to $2T_{n-1} + n$ which equals $L_n + n$ (by *Proposition VIII*), and this equals S_n (by *Proposition X*).

Proposition XIII. "Every square number plus the following odd number equals the following square number." (In symbols, $S_n + O_{n+1} = S_{n+1}$.)

Proposition XV. "The sum of the first n odd integers is equal to the nth square number."[2] (In symbols, $O_1 + O_2 + O_3 + \cdots + O_n = S_n$.) Maurolycus's proof freely translated is this:

[1] In *Proposition VII*, which I have not quoted, Maurolycus uses the usual arithmetical progression device for proving $T_n = [n(n + 1)]/2$. He says in effect: $T_n = 1 + 2 + 3 + \cdots + n$ and also $T_n = n + (n - 1) + (n - 2) + \cdots + 2 + 1$, and therefore by addition $2T_n = n(n + 1)$.

[2] This theorem in the form $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is often given in American college algebras as one of the first examples of complete induction to be proved by the student.

"By a previous proposition[1] the first square number (unity) added to the following odd number (3) makes the following square number (4); and this second square number (4) added to the 3d odd number (5) makes the 3d square number (9); and likewise the 3d square number (9) added to the 4th odd number (7) makes the 4th square number (16); and so successively to infinity the proposition is demonstrated by the repeated application of *Proposition XIII*."

This is a clear case of a complete induction proof. *Proposition XIII* is used as a lemma. It furnishes the argument from n to $n + 1$. The first few special cases are mentioned in *Proposition XV* itself. In modern symbols the proof would be this:

1st. The theorem is true when $n = 1$. *2d.* Assume that it is true when $n = k$, *i. e.*, assume $O_1 + O_2 + \cdots + O_k = S_k$; add O_{k+1} to both sides of this equation and get $O_1 + O_2 + \cdots + O_{k+1} = S_k + O_{k+1}$ which equals S_{k+1}, by *Proposition XIII*.

Pascal's Use of Complete Induction.

Pascal repeatedly used the method of complete induction in connection with his arithmetical triangle[2] and its applications. Cantor's argument that Pascal borrowed the method from Maurolycus is valid in spite of the fact that he is in error in saying that Maurolycus proved *Proposition XI* of his arithmetic by complete induction. For Pascal does refer to Maurolycus for a proof of this proposition which shows that Pascal was familiar with the part of Maurolycus's arithmetic in which Maurolycus does use complete induction. It is in a letter from Pascal (writing under the pseudonym Amos Dettonville) to Carcavi[3] that Pascal refers to Maurolycus for the proof of the theorem that "*twice the nth triangular number minus n equals* n^2." Pascal says "*Cela est aisé par Maurolic.*" Pascal makes use of this theorem in connection with his work on centers of gravity.

I give the following as two interesting examples of Pascal's use of the method of complete induction. I use modern algebraic notation for the sake of brevity. Without modern notation it would be necessary to explain the construction of Pascal's arithmetical triangle and many of his theorems about it.

Theorem.[4] *The number of combinations of n things k at a time is to the number of combinations of n things* $k + 1$ *at a time as* $k + 1$ *is to* $n - k$. (In symbols, ${}_nC_k : {}_nC_{k+1} = k + 1 : (n - k.)$

Proof.—First part. By inspection the theorem is true when $n = 2$. For then the only possible values of k and $k + 1$ are 1 and 2 respectively and ${}_2C_1 : {}_2C_2 = 2 : 1$.

Second part. Assume that the theorem is true when $n = q$. That is, assume[5]

$$(A) \qquad {}_qC_k : {}_qC_{k+1} = k + 1 : q - k$$

for all positive integral values of $k < q$. It is then to be proved that, on this assumption,

$$(B) \qquad {}_{q+1}C_j : {}_{q+1}C_{j+1} = j + 1 : q + 1 - j$$

for all positive integral values of $j < q + 1$.

[1] He refers to *Proposition XIII*.

[2] *Oeuvres Complètes de Blaise Pascal*, Paris, 1889, Vol. III, p. 243 ff.

[3] L. c., p. 376.

[4] This is Pascal's "Consequence XII," l. c., p. 248.

[5] This relation (A) is seen by inspection to be true when $k = q$, if we define ${}_sC_t = 0$ when $t > s$.

[(B) is obtained from (A) by replacing q in (A) by $q+1$ and by using another letter for k to avoid confusion later.] The well-known relation[1]

$$(C) \qquad {}_NC_R = {}_{N-1}C_{R-1} + {}_{N-1}C_R$$

is needed to prove that (B) follows from (A).

By relation (C) the left-hand member of (B) is equal to

$$\frac{{}_qC_{j-1} + {}_qC_j}{{}_qC_j + {}_qC_{j+1}} = \frac{\dfrac{{}_qC_{j-1}}{{}_qC_j} + 1}{1 + \dfrac{{}_qC_{j+1}}{{}_qC_j}}.$$

On applying relation (A) to the minor fractions ${}_qC_{j-1}/{}_qC_j$ and ${}_qC_{j+1}/{}_qC_j$ this becomes

$$\frac{\dfrac{j}{q-j+1} + 1}{1 + \dfrac{q-j}{j+1}} = \frac{j+1}{q-j+1}$$

which was to be proved.

Another example from Pascal is this problem in the division of stakes in a gambling game. Two players A and B of equal skill, playing for a stake P, wish to leave the game table before finishing their game. Their scores and the number of points which constitute the game being given indirectly as follows: Player A lacks α points of winning and player B lacks β points. If $\alpha + \beta$ be denoted by n, Pascal says that the stakes should be divided between B and A in the ratio

$$({}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{\alpha-1}) : ({}_{n-1}C_\alpha + {}_{n-1}C_{\alpha+1} + \cdots + {}_{n-1}C_{n-1}).$$

Since[2]

$${}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1},$$

this is the same as saying that B's share of the whole stake is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{\alpha-1}],$$

and A's share is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_\alpha + {}_{n-1}C_{\alpha+1} + \cdots + {}_{n-1}C_{n-1}].$$

Proof.—First Part. The theorem is true in the special case in which $n = 2$. For in this case the scores of A and B are even and each lacks only one point of winning. They are of equal skill and so one is as likely to win the game as the other and the stake should be divided in the ratio 1 to 1. The theorem states that the stake should be divided in the ratio ${}_1C_0 : {}_1C_1$, which is 1 : 1. The theorem is also true in the special case in which $n = 3$. In this case the score at the end of play must be such that one player lacks one point and the other lacks two points. Suppose that it is A who lacks the one point. Then B lacks two points. If the play were to continue for one more point and if A were to win that point he would win the game and be entitled to the whole stake P. But if he were to lose he would be entitled to $P/2$ by virtue of the special

[1] Fine, H. B., *College Algebra*, p. 404. This relation is true when $R = 0$ and when $N \geqq R$ if we define ${}_sC_0 = 1$ and ${}_sC_t = 0$ when $t > s$.

[2] It is customary to give ${}_NC_0$ the meaning ${}_NC_0 = 1$ by definition. The relation ${}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1}$ is more usually written ${}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1} - 1$. See Fine, H. B., *College Algebra*, p. 402.

case previously considered. The division of the stake should therefore be such that A will get the $P/2$ which he is entitled to in case he loses the next point and one half of the other $P/2$ which he has an even chance of winning. This is the same as saying that A should receive the arithmetic mean between P and $P/2$. The stake should therefore be divided between B and A in the ratio 1 to 3. The theorem gives the ratio ${}_2C_0 : ({}_2C_1 + {}_2C_2)$, which is $1 : 3$.

If A lacked 2 points and B one point the division between B and A would be in the ratio $3 : 1$. The theorem in this case gives $({}_2C_0 + {}_2C_1) : {}_2C_2$ which is $3 : 1$.

Second Part of the Proof. Assume the theorem true for n, *i. e.*, assume that when the points lacking for A and B have their sum equal to n the equitable division of the stake is as the theorem indicates. To prove that on this assumption the theorem is true when the sum of the points lacking for A and B is $n + 1$, suppose that A lacks k points and B lacks l points where $k + l = n + 1$. If the play were to continue and A were to win the next point, the sum of the points lacking after that would be exactly n points (*i. e.*, A would lack $k - 1$ points and B would lack l points) and by our assumption the rule given by the theorem may be applied. The result of applying the theorem in this case is

$$\frac{P}{2^{n-1}}\left[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}\right]$$

for B's share of the stake. (This comes from putting $k - 1$ for α in the statement of the theorem.) But if A were to lose then B would win and A would lack k points and B only $l - 1$ points. The sum of the points lacking would be exactly n and the rule given by the theorem may be applied as before, putting k for α. The result is

$$\frac{P}{2^{n-1}}\left[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-1}\right]$$

for B's share. As in the special case previously considered B's share of the stake should be the arithmetic mean between

$$\frac{P}{2^{n-1}}\left[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}\right] \quad \text{and} \quad \frac{P}{2^{n-1}}\left[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-1}\right]$$

which equals

$$\frac{P}{2^n}\left[2({}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}) + {}_{n-1}C_{k-1}\right].$$

This may be written in the form

$$\frac{P}{2^n}\left[{}_{n-1}C_0 + ({}_{n-1}C_0 + {}_{n-1}C_1) + ({}_{n-1}C_1 + {}_{n-1}C_2) + \cdots + ({}_{n-1}C_{k-3} + {}_{n-1}C_{k-2}) + ({}_{n-1}C_{k-2} + {}_{n-1}C_{k-1})\right].$$

But by virtue of the relation ${}_NC_R = {}_{N-1}C_{R-1} + {}_{N-1}C_R$, used in the proof of the preceding theorem of Pascal's, each binomial in this expression may be replaced by a single term so that B's share is

$$\frac{P}{2^n}\left[{}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{k-2} + {}_nC_{k-1}\right]$$

which is just what the theorem gives for B's share.

This completes the induction argument for this theorem of Pascal's.

Other and More Recent Uses of Complete Induction.

The following examples will give some idea of the variety of uses which can be made of the method of complete induction.

1. (*a*) $x^n - y^n$ *is divisible by* $x - y$.
   (*b*) $x^n - y^n$ *is divisible by* $x + y$ *when n is even but not*[1] *when n is odd.*
   (*c*) $x^n + y^n$ *is divisible by* $x + y$ *when n is odd but not*[1] *when n is even.*

[1] The reference here (as elsewhere in the paper) is to *algebraic* divisibility. The statements obviously need not be true for the case of divisibility of the integers represented by the forms.

Negative divisibility theorems like the second parts of (*b*) and (*c*) are just as easily proved by complete induction as positive divisibility theorems such as (*a*).

2. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

American college algebras contain many such examples in the summation of series.

3. *De Moivre's Theorem.* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ *for positive integral values of* n.

4. *The binomial theorem for positive integral exponents.*

5. *If*
$$F_n(x) \equiv (x+a_1)(x+a_2)(x+a_3)\cdots(x+a_n)$$
$$\equiv x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n,$$
then
$$A_1 = \Sigma a_1, \quad A_2 = \Sigma a_1a_2, \quad A_3 = \Sigma a_1a_2a_3, \cdots, \quad A_n = a_1a_2\cdots a_n.$$

(See H. Weber and J. Wellstein, *Encyclopädie der Elementar-Mathematik*, volume I, p. 196.)

6. *Fermat's Theorem.* $a^n - a$ *is divisible by* n *if* a *is any integer and* n *is a prime integer.* (See Weber and Wellstein, volume I, p. 197.)

7. *The Polynomial Theorem.*
$$(x+y+z+\cdots)^n = \Sigma\frac{n!}{\alpha!\beta!\gamma!\cdots}x^\alpha y^\beta z^\gamma\cdots.$$

(See Weber and Wellstein, volume I, p. 198.)

8. *Any polynomial symmetric in* $x_1, x_2, \cdots, x_n$ *is equal to a polynomial in the elementary symmetric functions.* (See Weber and Wellstein, volume I, p. 232.)

9. *A necessary and sufficient condition that a polynomial in any number of variables vanish identically is that all its coefficients are zero.* (For a proof of this theorem for one variable and its extension by complete induction see Maxime Bôcher's *Introduction to Higher Algebra*, p. 5.)

10. *If* f_1 *and* f_2 *are polynomials in any number of variables of degrees* m_1 *and* m_2 *respectively, the product* f_1f_2 *will be of degree* m_1m_2. (See Bôcher's *Higher Algebra*, p. 6.)

11. *A necessary and sufficient condition that a polynomial* $f(x_1x_2\cdots x_n)$ *vanish identically is that it vanish throughout the neighborhood of a point* $(a_1a_2\cdots a_n)$. (See Bôcher's *Higher Algebra*, p. 10.)

12. *If* $f(x)$ *is a polynomial of the nth degree,*
$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \qquad (a_0 \neq 0),$$
there exists one and only one set of constants $\alpha_1\alpha_2\cdots\alpha_n$ *such that*
$$f(x) \equiv a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

To prove this theorem by complete induction one needs the fundamental theorem of algebra that there is at least one value of x for which such a polynomial as $f(x)$ vanishes. (See Bôcher's *Higher Algebra*, p. 17.)

13. *If all the* $(r+1)$*-rowed principal minors of the symmetrical matrix*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (\text{where } a_{ij} = a_{ji})$$

are zero, and also all the $(r+2)$*-rowed principal minors, then the rank of the matrix is* r *or less.* (See Bôcher's *Higher Algebra*, p. 57.)

14. *The conjugate of the product of any number of matrices is the product of their conjugates taken in reverse order.* (See Bôcher's *Higher Algebra*, p. 65.)

15. *If* $y = u_1u_2\cdots u_n$ *and* y', u_1', u_2', *etc., denote first derivatives with respect to a variable* x, *then*

$$\frac{y'}{y} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}.$$

16. *Leibnitz's Theorem.*

$$\frac{d^n(uv)}{dx^n} = u_nv + nu_{n-1}v_1 + \frac{n(n-1)}{1\cdot 2}u_{n-2}v^2 + \cdots + nu_1v_{n-1} + uv_n,$$

where u *and* v *are functions of* x *and the subscripts denote derivatives with respect to* x. *That is,*

$$u_1 = \frac{du}{dx}, \qquad u_2 = \frac{d^2u}{dx^2}, \qquad \text{etc.}$$

(For a proof of this theorem and some examples of its use see G. A. Osborne's *Differential and Integral Calculus*, pages 65–67.)

17. (*a*) *Limit* $(x_1 + x_2\cdots + x_n) = $ *limit* $x_1 +$ *limit* $x_2 + \cdots +$ *limit* x_n.

(*b*) *Limit* $(x_1x_2\cdots x_n) = (\text{limit } x_1)(\text{limit } x_2)\cdots(\text{limit } x_n)$.

(See W. F. Osgood's *Differential and Integral Calculus*, p. 16.)

18. *If*

$$y = \log x, \qquad \frac{d^ny}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Numerous examples like this are to be found in G. A. Osborne's *Differential Calculus*, pages 62–65.

QUADRATIC FACTORS OF POLYNOMIALS.

By A. F. FRUMVELLER, Marquette University.

All methods of finding quadratic factors of $f(x) = \sum_0^n (a_i x^i)$ will be empirical to a certain extent; if $x^2 + mx + n$ be the factor sought, any one of the divisors of a_0 is taken as n, and its associated number m remains to be found. The older method[1] (quoted, for example, in Wentworth, McLellan and Glashan's "Algebraic Analysis") consisted in arranging in a vertical column the expressions

$$\cdots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \cdots$$

and placing after each of these numbers all its factors written with both plus and minus signs. If $r_i^{(s)}$ be one of these factors, lying in the horizontal row of $f(s)$, add s^2 to it. Then the numbers $r_i^{(s)} + s^2$ thus formed are kept on the line of $f(s)$, and constitute our actual working-table. If now an arithmetic sequence can be discovered running *vertically* through this table, the *difference* of this sequence is a provisional value of m, to be tried out by synthetic division. It is most laborious and unsatisfactory to attempt this scheme.

The method proposed by Professor Glenn, in this MONTHLY for October, 1916, locates m at the outset among the factors of a certain symmetric function of the coefficients, namely

$$P(x_i + x_j), \quad (i \neq j;\ i, j = 1, 2, 3, \cdots n);$$

the proper association of each m with its own n being then determined experimentally by synthetic division. He uses the upper limit L of the roots to cut down the number of the possible combinations, by excluding values of (m, n) that exceed $(2L, L^2)$. This is a great improvement; but there is still too much room for experiment, and the real problem of getting m as a $f(n)$ is not met. The ideal solution would be to assign a process or an equation which would directly associate with each assumed n its own m, or, at least, a minimum number of m's from which to choose. The methods presented herewith for that purpose seem to be new, and may be of interest on account of their rapid and easy application.

The Quartic. Let

$$f(x) \equiv x^4 + ax^3 + bx^2 + cx + d = (x^2 + mx + n)(x^2 + px + q).$$

Then (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq$, (4) $d = nq$. From (1) and (3), we get $m = (c - an)/(q - n)$, which from the nature of the problem must be integral. We now try every $n < L^2$, and check the m's found in (2), which, combining with (1), becomes

$$m^2 - am - (n + q - b) = 0.$$

Thus, for $\left\{\begin{matrix} a & b & c & d \\ 1 & -6 & +3 & +22 & -6 \end{matrix}\right\}$, we find $(m, n) = (*, -1), (-4, 1),$

[1] See also Kronecker, Runge and Mandl in *Crelle's Journal*, Vols. 92, 99 and 113 respectively.

$\cdots$ and this last satisfies (2). Hence $x^2 - 4x + 1$ is a factor. Again, using $\left\{\begin{matrix} a & b & c & d \\ 1 & +4 & -4 & -17 & +10 \end{matrix}\right\}$ (see Glenn, loc. cit.), we have $(m, n) = (*, 1)$, $(*, -1)$, $(*, 2)$, $(3, -2)$. This last pair satisfies (2), hence

$$f(x) = (x^2 + 3x - 2)(x^2 + x - 5).$$

For brevity and directness, this method leaves little to be desired.

The Quintic. Let

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = (x^2 + mx + n)(x^3 + px^2 + qx + r).$$

Then (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq + r$, (4) $d = nq + mr$, (5) $e = nr$. If we find q from (4) and (5), and also from (1) and (2), and compare these values, we get

$$m^2 + m\left(\frac{e}{n^2} - a\right) + \left(b - n - \frac{d}{n}\right) = 0,$$

where m is an integer. For each m found, compute p and q, and check in (3).

Using $\left\{\begin{matrix} a & b & c & d & e \\ 1 & -1 & 6 & 9 & 1 & -4 \end{matrix}\right\}$, we find for $n = \pm 1, \pm 2$, m turns out fractional or imaginary; hence there is no quadratic factor.

Again, using $\left\{\begin{matrix} a & b & c & d & e \\ 1 & -1 & -6 & 9 & 1 & -4 \end{matrix}\right\}$, for (m, n) we get $(*, 1)$, $(4, -1)$, $(-1, -1)$; and on computing p and q, the check is satisfied for $m = -1$, $n = -1$, $p = 0$, $q = -5$. Hence $f(x) = (x^2 - x - 1)(x^3 - 5x + 4)$.

The Sextic. The conditional equations are: (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq + r$, (4) $d = nq + mr + s$, (5) $e = nr + ms$, (6) $f = ns$. Hence

$$q = \frac{1}{n^3}\begin{vmatrix} d & m & 1 \\ e & n & m \\ f & 0 & n \end{vmatrix} = \begin{vmatrix} 1 & a - m \\ m & b - n \end{vmatrix},$$

and the m-eliminant is, therefore,

$$m^2\left(\frac{f}{n} - n^2\right) + m(an^2 - e) + (n^3 - n^2b + dn - f) = 0.$$

Using $\left\{\begin{matrix} a & b & c & d & e & f \\ 1 & 2 & -5 & -7 & 8 & 3 & -2 \end{matrix}\right\}$, with (3) as a check, for $n = -1$ we have $m = 2$, or -1; for $n = -2$, $m = 1$. These numbers give the three quadratic factors.

It will be noticed that the m-eliminant is a quadratic for the *quintic* and *sextic;* a cubic for the *septic* and *octic;* for a polynomial of degree $2k$ or $2k - 1$, it is of degree $k - 1$. The use of determinants makes it easy to write out these equations. One check-equation will always be left over, by means of which the

$(k-1)$ m-values can be tested. Having the m-equation, we therefore insert $n < L^2$, and look for integral roots by synthetic division, as far as $m < 2L$, where L is the upper (lower) limit of the roots of $f(x)$. We are then sure that the m's thus found are correctly paired with their own proper n.

The following table gives the m-eliminants for degrees 4 to 9; the law of their structure is obvious, so that the table can be extended indefinitely.

4-ic: $\frac{1}{n^2}\begin{vmatrix} c & m \\ d & n \end{vmatrix} - |a - m|$ $= 0$. Check in the b-equation.

5-ic: $\frac{1}{n^2}\begin{vmatrix} d & m \\ e & n \end{vmatrix} - \begin{vmatrix} 1 & a-m \\ m & b-n \end{vmatrix}$ $= 0$. Check in the c-equation.

6-ic: $\frac{1}{n^3}\begin{vmatrix} d & m & 1 \\ e & n & m \\ f & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & a-m \\ m & b-n \end{vmatrix}$ $= 0$. Check in the c-equation.

7-ic: $\frac{1}{n^3}\begin{vmatrix} e & m & 1 \\ f & n & m \\ g & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & a-m \\ m & 1 & b-n \\ n & m & c \end{vmatrix}$ $= 0$. Check in the d-equation.

8-ic: $\frac{1}{n^4}\begin{vmatrix} e & m & 1 & 0 \\ f & n & m & 1 \\ g & 0 & n & m \\ h & 0 & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & a-m \\ m & 1 & b-n \\ n & m & c \end{vmatrix}$ $= 0$. Check in the d-equation.

9-ic: $\frac{1}{n^4}\begin{vmatrix} f & m & 1 & 0 \\ g & n & m & 1 \\ h & 0 & n & m \\ i & 0 & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & a-m \\ m & 1 & 0 & b-n \\ n & m & 1 & c \\ 0 & n & m & d \end{vmatrix}$ $= 0$. Check in the e-equation.

The General Polynomial. To find an explicit form of the m-equation for the general case, it will be expedient to use a more systematic notation, and to modify the preceding method somewhat, so as to avoid having two determinants in the same equation. Let

$$f(x) \equiv x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0$$
$$= (x^2 + mx + n)(x^{k-2} + b_{k-3}x^{k-3} + \cdots + b_0).$$

Performing the multiplication and equating coefficients, we get

$$\begin{aligned} a_{k-1} &= m + b_{k-3}, \\ a_{k-2} &= n + mb_{k-3} + b_{k-4}, \\ a_{k-3} &= nb_{k-3} + mb_{k-4} + b_{k-5}, \\ &\cdots\cdots\cdots\cdots \\ a_1 &= nb_1 + mb_0, \\ a_0 &= nb_0. \end{aligned}$$

This is a consistent set of k linear equations between the $(k-1)$ quantities $[1, b_{k-3}, b_{k-4}, \cdots, b_1, b_0]$. Eliminating these, we have at once as our m-equation

(I)
$$0 = \begin{vmatrix} a_{k-1} & m & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{k-2} & n & m & 1 & 0 & \cdots & 0 & 0 \\ a_{k-3} & 0 & n & m & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & 0 & 0 & 0 & 0 & \cdots & n & m \\ a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & n \end{vmatrix}.$$

This is of the $(k-1)$th degree in m, whereas the equations formerly obtained were only of degree $(k-3)/2$ or $(k-4)/2$, so that the number of m's belonging to a given n is doubled; but the present equation is more readily handled for the purpose we have in view, namely, to display the inner structure of the m-eliminant.

Let us use the symbols

$$\Delta_2, \Delta_3, \cdots \text{ for } \begin{vmatrix} m & 1 \\ n & m \end{vmatrix}, \begin{vmatrix} m & 1 & 0 \\ n & m & 1 \\ 0 & n & m \end{vmatrix}, \cdots.$$

Then, on continuously developing (I) in terms of its last row, we get

(II)
$$\begin{aligned} 0 &= a_0\Delta_{k-1} - na_1\Delta_{k-2} + \cdots + (-1)^{k-1}n^{k-1}a_{k-1}\Delta_0 \\ &= \sum_{\lambda=0}^{\lambda=k-1} [(-1)^\lambda n^\lambda a_\lambda \Delta_{k-\lambda-1}]; \quad (\Delta_1 = m;\ \Delta_0 = 1). \end{aligned}$$

It remains to evaluate Δ_r. Let us write down a few of the first determinants of this form, to observe the law, noting that $\Delta_r = m\Delta_{r-1} - n\Delta_{r-2}$.

$$\Delta_1 = m, \qquad \Delta_5 = m^5 - 4m^3n + 3mn^2,$$

$$\Delta_2 = m^2 - 1n, \qquad \Delta_6 = m^6 - 5m^4n + 6m^2n^2 - 1n^3,$$

$$\Delta_3 = m^3 - 2mn, \qquad \Delta_7 = m^7 - 6m^5n + 10m^3n^2 - 4mn^3,$$

$$\Delta_4 = m^4 - 3m^2n + 1n^2, \qquad \Delta_8 = m^8 - 7m^6n + 15m^4n^2 - 10m^2n^3 + 1n^4.$$

If one desires an explicit general formula for Δ_r it may be found as follows: The above coefficients seem to be arithmetic sequences of order 0, 1, 2, 3, $\cdots$ and a simple induction shows that this is really so. If we let $T_q^{(j)}$ stand for the *qth term of an arithmetic sequence of order j*, we have

$$\begin{cases} \Delta_{2h} = m^{2h} - T_{2h-1}^{(1)}m^{2h-2}n + \cdots + (-1)^h T_1^{(h)} m^0 n^h, \\ \Delta_{2h+1} = m^{2h+1} - T_{2h}^{(1)}m^{2h-1}n + \cdots + (-1)^h T_2^{(h)} mn^h; \end{cases}$$

or more briefly,

(III)
$$\Delta_r = \sum_{j=0}^{j=h} [(-1)^j T_{r-2j+1}^{(j)} m^{r-2j} n^j], \quad (r = 2h, \text{ or } 2h+1).$$

The T's can be found from the following table:

0th order:	1	1	1	1	1	1	1	1	1	1	...
1st " :		1	2	3	4	5	6	7	8		...
2d " :			1	3	6	10	15	21	28		...
3d " :				1	4	10	20	35	56		...
4th " :					1	5	15	35	70	126	...
5th " :						.	.	.	.	.	...

The formula for the qth term in the line j is[1]

$$T_q^{(j)} = d_1 + C_1^{q-1}d_2 + C_2^{q-1}d_3 + \cdots + C_j^{q-1}d_j.$$

In our case, the d's themselves are binomial coefficients of order j, so that

$$T_q^{(j)} = \sum_{i=0}^{i=j} (C_i^{q-1}C_i^j),$$

where $C_0^s = C_0^0 = 1$, and $C_k^s = 0$, $(k > s)$. Inserting this in (III), we obtain

$$\Delta_r = \sum_{j=0}^{j=h} \left((-1)^j \sum_{i=0}^{i=j} [C_i^{r-2j}C_i^j] m^{r-2j} n^j \right), \quad (r = 2h, \text{ or } 2h+1).$$

Finally we make use of the proposition that $T_q^{(j)} = H_j^q = C_j^{q+j-1}$, so that

$$C_s^{n-s} = \sum_{i=0}^{i=s} [C_i^{n-2s}C_i^s],$$

and get as our definite result

$$\text{(IV)} \qquad \Delta_r = \sum_{j=0}^{j=h} ((-1)^j C_j^{r-j} m^{r-2j} n^j).$$

Equations (II) and (IV) completely solve the problem of associating the n of a quadratic factor of $f(x)$ with its own proper m's, and display the nature of this association.

Incidentally, we get in (IV) the expansion of an interesting determinant. If we replace 1 by r, and put $\Delta_q \equiv D_q(n, m, r)$, we have

$$D_q(n, m, r) = \sum_{j=0}^{j=h} [(-1)^j C_j^{q-j} m^{q-2j} n^j r^j].$$

Among the special forms, we note $D_q(n, m, -1)$, which is a continuant, and $D_q(1, \pm 1, \pm 1)$, which gives us curious summation formulas.

[1] Fine, *College Algebra*, p. 365.

ON THE CONSTRUCTION OF CERTAIN CURVES GIVEN IN POLAR COÖRDINATES.

By R. E. MORITZ, University of Washington.

1. *Definitions.* Consider the parametric equations in cartesian coördinates,

$$y = a \cos pt + k, \tag{1}$$

$$x = qt. \tag{2}$$

(1) represents a simple harmonic motion, t the time, y the distance of the vibrating point from a fixed point in the line along which the vibration takes place, a the amplitude and $2\pi/p$ the period of the vibration, k the distance of the mean point of vibration from the fixed point. (2) represents a uniform linear (translatory) motion at right angles to the direction of the line along which the simple harmonic motion takes place, q the uniform velocity of the point, x its distance from a fixed point at any given time t. (1) and (2) taken simultaneously represent the motion resulting from the composition of these two motions. This resultant motion we shall call a *translatory-harmonic* motion, the locus of this motion the *linearly-harmonic* curve, whose equation

$$y = a \cos \frac{p}{q} x + k \tag{3}$$

is obtained by eliminating the parameter t from the component equations (1) and (2).

Definition 1. A translatory-harmonic motion ($y = a \cos pt + k$, $x = qt$) *is the motion of a point which has simple harmonic motion* ($y = a \cos pt + k$) *along a line, while at the same time the line moves with a constant velocity* q *at right angles to itself. The locus of the resultant motion is the linearly-harmonic curve* $y = a \cos (p/q)x + k$.

Let us now consider the parametric equations in polar coördinates,

$$\rho = a \cos pt + k, \tag{4}$$

$$\theta = qt. \tag{5}$$

Like (1), (4) represents a simple harmonic motion, ρ being the distance of the vibrating point from a fixed point in the line along which the vibration takes place, while a, p, t and k have the same meaning as in (1). (5) represents a uniform angular (rotatory) motion, q the uniform angular velocity, θ the vectorial angle of the rotating point at any given time t. (4) and (5) taken simultaneously represent the motion resulting from the composition of these two motions. This resultant motion we shall call a *rotatory-harmonic* motion, the locus of this motion is the *cyclic-harmonic* curve

$$\rho = a \cos \frac{p}{q} \theta + k, \tag{6}$$

obtained by eliminating the parameter t from the component equations (4) and (5).

Definition 2. A rotatory-harmonic motion ($\rho = a \cos pt + k$, $\theta = qt$) *is the motion of a point which has simple harmonic motion* ($\rho = a \cos pt + k$) *along a line, while at the same time this line rotates with a constant angular velocity q about one of its fixed points. The locus of the resultant motion is the cyclic-harmonic curve* $\rho = a \cos (p/q)\theta + k$.

2. *Construction of Cyclic-harmonic Curves by Points.* The foregoing definition furnishes a convenient method of constructing by points any cyclic-harmonic curve whose equation is given.

Let the equation of the curve be $\rho = a \cos (p/q)\theta + k$.

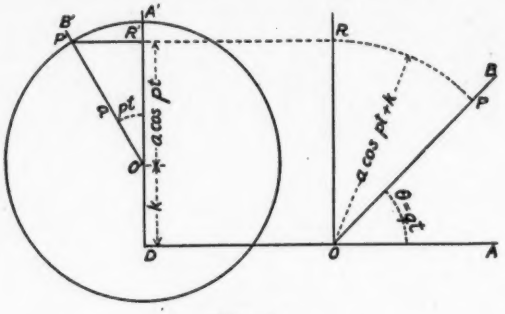


FIG. 1.

Let O (Fig. 1) represent the origin of coördinates, OA the initial line. At a point D on AO produced, chosen at any convenient distance from O, construct a perpendicular DA' and on it take DO' equal to k. Select any convenient unit of angular measure and construct the AOB ($= \theta$) and $A'O'B'$ ($= \theta'$) equal to qt and pt units respectively, t being any arbitrarily assumed integer. On $O'B'$ take $O'P'$ equal to a and let R' be the projection of P' on DA'. With O as a center and DR' as a radius describe an arc. Then P, the intersection point of this arc with OB in case DR' is positive, or with BO produced in case DR' is negative, that is in case R' falls below D, is a point on the required curve; for in either case

$$\rho = OP = OR = DR' = DO' + O'R' = k + a \cos pt, \quad \theta = qt,$$

from which on eliminating t, $\rho = a \cos (p/q)\theta + k$.

By choosing the unit of angular measure sufficiently small, and taking in turn $t = 0, 1, 2, 3$, etc., we may thus construct as many points of the curve as desired, at intervals small at will. Figs. 2, 3, 4, 5, show the method applied to the construction of the cyclic-harmonics $\rho = a \cos \frac{3}{2}\theta + k$, for the values $k = 3a$, $k = a$, $k = a/2$, $k = 0$, respectively.

3. *Classification of Cyclic-harmonic Curves.* Let the ratio p/q determine the species of the curve $\rho = a \cos (p/q)\theta + k$. An inspection of the preceding figures

discloses certain properties which are independent of the particular value of the ratio p/q employed and which are therefore common to all the species. Fig. 2 has an open center and it follows from the mode of construction, as is otherwise obvious from the form of the equation, that the curve is confined between two circles whose radii are $k - a$ and $k + a$ respectively. Fig. 3 consists of leaves which meet in cusps at the center, the axial diameter of each leaf is $k + a$. Fig. 4 consists of two sets of leaves, the axial diameters of the larger set being $k + a$, those of the smaller set $k - a$. Fig. 5 consists of a single set of equal leaves whose axial diameters equal a.

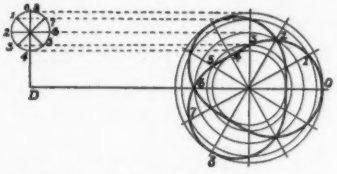


FIG. 2. Curtate Cyclic-harmonic,

$$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = 3.$$

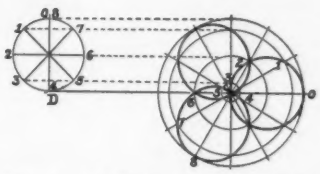


FIG. 3. Cuspitate Cyclic-harmonic

$$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = 1.$$

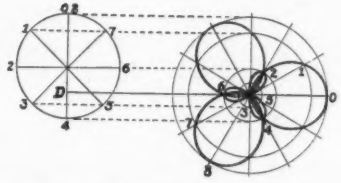


FIG. 4. Prolate Cyclic-harmonic,

$$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = \frac{1}{2}.$$

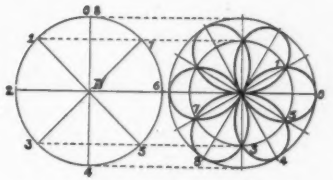


FIG. 5. Foliate Cyclic-harmonic,

$$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = 0.$$

These properties serve as a convenient basis for the further classification of the cyclic-harmonic curves of any species. We shall call a cyclic-harmonic curtate if $k > a$, cuspitate if $k = a$, prolate if $0 < k < a$, equi-foliate (or foliate) if $k = 0$.

The cyclic-harmonic curves embrace as special cases a considerable number of familiar curves. Pascal's conchoids constitute one species, $p/q = 1$. The cardioid and Münger's double egg curves are cuspitate cyclic-harmonics, $p/q = 1$ and 2 respectively. The common limaçon and Freeth's nephroid are prolate cyclic-harmonics, $p/q = 1$ and $\frac{1}{2}$ respectively. All roses (Rosenkurven, rosaces) are equi-foliate cyclic-harmonics. The linearly harmonic (simple harmonic) curves will be shown to be degenerate forms of curtate cyclic-harmonic curves.

4. *The Cyclo-harmonograph.* The foregoing definition of rotatory harmonic motion suggests a simple mechanism for constructing any cyclic-harmonic curve kinematically.

A wheel W_1, center C, carries a crank-pin R which, as the wheel rotates, slides in a slotted cross-bar ST of a cross-head HK perpendicular to ST. This cross-head is constrained to move in the direction of HK by means of two fixed guides F and G. As the wheel W_1 moves with constant angular velocity about its center C, any point P in HK will have simple harmonic motion in the direction of HK.

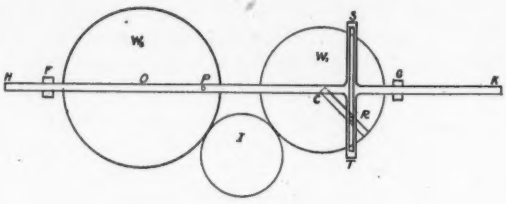


FIG. 6.

Let now the mechanism thus far described be made to revolve about a fixed wheel W_2, center O, by means of an idle-wheel I which connects the circumferences of the wheels W_1 and W_2. Any point P in HK will then receive rotatory harmonic motion and a pencil or pen-point placed at P will trace out a cyclic-harmonic curve in the plane of the paper.

To deduce the equation of the curve traced out by P, let the dotted lines in Fig. 7 represent the initial position of the mechanism, which is so chosen that the crank-pin R_0 is in the line of centers OC_0 of the wheels W_2 and W_1 and such that C_0 is between O and R_0.

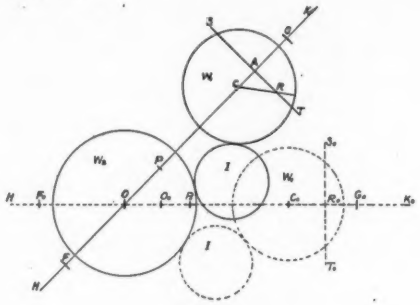


FIG. 7.

Take $R_0O_0 = C_0O$, the distance between the centers of the two wheels W_1 and W_2, and let k denote the distance of any point P_0 on H_0K_0 from O_0.

Let C represent the center of the wheel W_1 in any other position, $P\ R\ H\ K\ F\ G\ S\ T$ the corresponding positions of $P_0\ R_0\ H_0\ K_0\ F_0\ G_0\ S_0\ T_0$ respectively, and A the intersection of ST with HK.

Take O for the origin of coördinates, OK_0 for the initial line, and denote the coördinates of the point P by ρ, θ. Furthermore let $C_0R_0 = CR = a$, and

angle $RCK = \phi$. Then

$$OP = \rho = OC + CA - PA, \quad OC = OC_0 = O_0R_0,$$

$$CA = CR\cos\phi = a\cos\phi, \quad PA = P_0R_0 = O_0R_0 - k.$$

Hence

$$OP = \rho = O_0R_0 + a\cos\phi - (O_0R_0 - k), \text{ or } \rho = a\cos\phi + k.$$

Now let p and q represent the radii of the wheels W_2 and W_1 respectively, then obviously $q\phi = p\theta$, $\phi = (p/q)\theta$, and we have

$$\rho = a\cos\frac{p}{q}\theta + k \tag{7}$$

as the general equation of the locus of the point P.

The form of equation (7) shows that the locus of P is independent of the distance between the centers of the wheels W_1 and W_2, but depends only on the dimensions of the two wheels, the distance of the crank-pin R from the center C and the arbitrary distance O_0R_0. The single wheels W_1 and W_2 may therefore be replaced by two trains of wheels of various diameters, the idle-wheel I being used to connect at will any wheel of either train with any wheel of the other. A sliding carriage on HK and a crank-pin adjustable to various distances CR makes it possible to assign to k and c arbitrary values within the physical limits of the mechanism. Finally, it is immaterial whether the wheel W_2 is kept fixed while W_1 revolves about it, or whether the centers of both wheels remain fixed and both wheels be allowed to revolve, the paper on which the curve is traced being attached to the face of the wheel W_2. For practical reasons the latter is the more advantageous arrangement.

5. *Linearly-harmonic Curves as Degenerate Forms of Cyclic-harmonics.* Fig. 8 shows a portion of a curtate cyclic-harmonic curve. Plainly, as the arc which

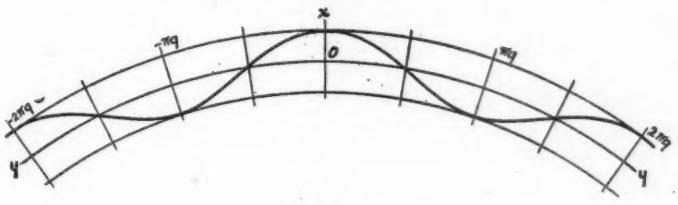


FIG. 8.

forms the directrix of this curve degenerates into a straight line the curve itself will degenerate into a linearly harmonic curve.

In Fig. 7 let x, y denote the rectangular coördinates of any point ρ, θ of the cyclic-harmonic curve $\rho = a\cos(p/q)\theta + k$, referred to C_0 as a new origin with C_0K_0 as the positive direction of the new x-axis. Put $k = l = OC_0$, the distance between the centers of the wheels W_2 and W_1. We then have

$$x + l = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad \rho = \sqrt{(x+l)^2 + y^2},$$

$$\frac{p}{q}\theta = \cos^{-1}\frac{\rho - l}{a} = \frac{p}{q}\tan^{-1}\frac{y}{x+l}. \tag{8}$$

Now let p increase indefinitely while q remains constant, then since $l = p + q + a$ constant, l also will increase indefinitely, and it is easy to see that

$$\lim_{l=\infty}\frac{\rho - l}{a} = \frac{x}{a}, \qquad \lim_{l=\infty}\left[\frac{p}{q}\tan^{-1}\frac{y}{x+l}\right] = \frac{y}{q},$$

hence equation (8) becomes in the limit

$$\cos^{-1}\frac{x}{a} = \frac{y}{q},$$

or

$$x = a\cos\frac{y}{q},$$

the equation of a linearly-harmonic curve, amplitude a, wave-length $2\pi q$.

6. *The Division of the Circle into Any Number of Parts.* The cyclo-harmonograph effects a complete solution of the classic problem of cyclotomy by determining the vertices of a regular polygon of any odd number of sides. It is easily shown that the cyclic harmonic curve $\rho = a\cos(p/q)\theta + k$ has a set of $p(q-1)$ nodes corresponding to the angles

$$\theta_{\lambda,\mu} = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \lambda = 1, 2, 3, \cdots, q-1; \qquad \mu = 0, 1, 2, \cdots, p-1.$$

These nodes lie in sets of $q-1$ on the p straight lines $\theta_\lambda = (\lambda/p)\pi$, in which $\lambda = 0, 1, 2, \cdots, p-1$, and in sets of p on the $q-1$ circles $\rho_\mu = a\cos(\mu/q)\pi + k$, in which $\mu = 1, 2, 3, \cdots, q-1$. It is clear therefore that any set of these nodes situated on the same circle determines the vertices of a regular polygon with n sides. In particular the cyclic-harmonic $\rho = a\cos(p/2)\pi + k$ has the p nodes $\theta_\lambda = (\lambda/p)\pi$, in which $\rho_\lambda = k$, $\lambda = 1, 3, 5, \cdots, 2p-1$.

7. *The Number of Species.* The number of species of cyclic-harmonics which it is possible to describe with a given cyclo-harmonograph depends of course on the number of wheels in the train of gears employed in the mechanism. Suppose that there are n wheels in each train and that the diameters of these wheels are proportional to the numbers $1, 2, 3, \cdots, n$. Let the diameters of the two wheels which the idle-wheel connects be as p is to q. Assume p constant and greater than q, with this assumption there are $\phi(p)$ admissible values of the ratio p/q, $\phi(p)$ being the totient function of p, that is, the number of numbers which are less than p and have with it no common divisor other than unity. Now let p take all values from 1 to n and we obtain $\Sigma_1^n \phi(n)$ species of curves under the restriction $n \geqq p > q$. Evidently there is an equal number of species under the restriction $n \geqq q > p$. Besides these there is the case $p/q = 1$. The total

number of species within the range of a cyclo-harmonograph having n wheels in each train of gears is therefore $1 + 2\,\Sigma_1^n\,\phi(n)$.

8. *Explanation of Plates.* The plates which follow contain four each of six species of cyclo-harmonic curves corresponding to the values $p/q = 2,\ 1/2,\ 9/2,\ 2/9,\ 8/5$, and $10/9$ respectively. Of each set of four, the first is curtate, the second cuspitate, the third prolate, and the fourth equi-foliate. The pole is taken at the center of each figure, the polar axis extending to the right. In the cases $k = 0$, k has actually been chosen slightly different from zero in order to exhibit more clearly the cusps at the origin.

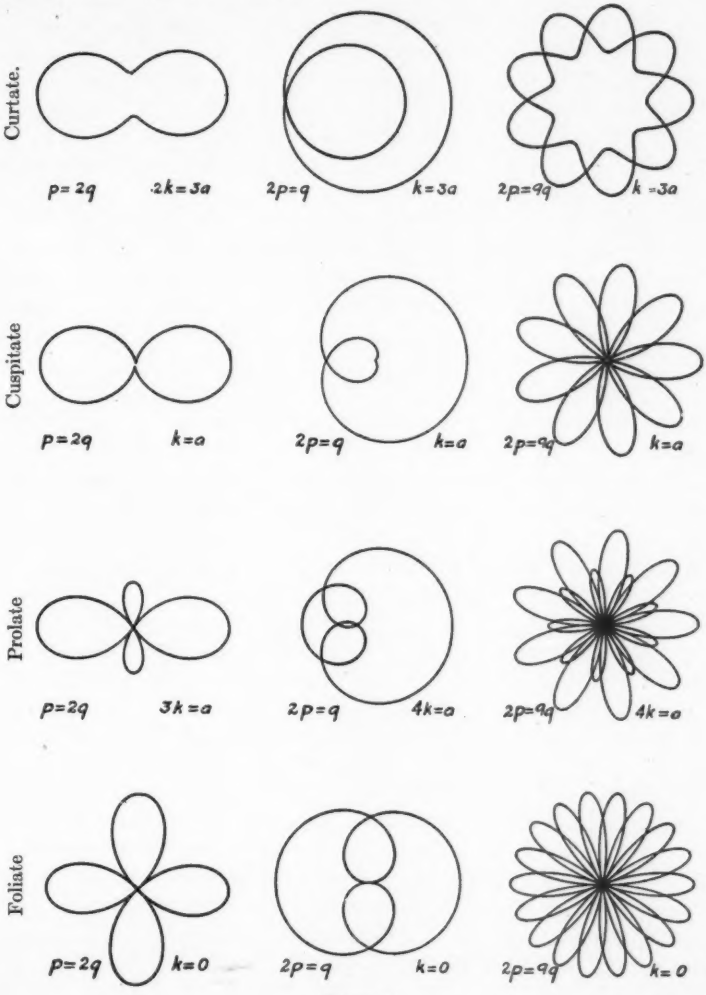


PLATE I.

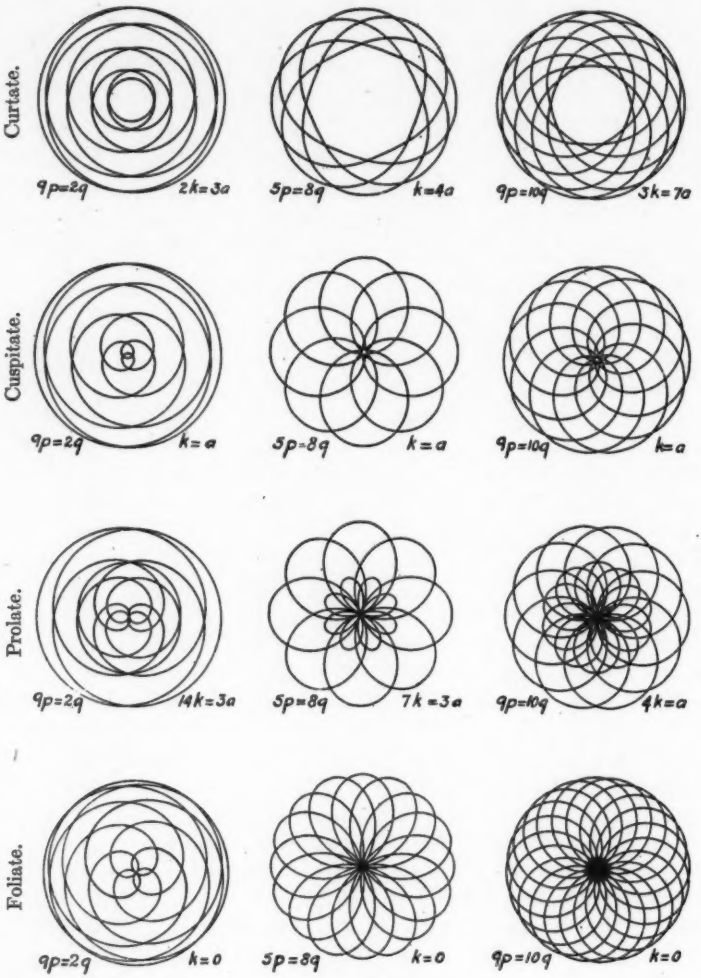

Curtate.
9p=2q
2k=3a
5p=8q
k=4a
9p=10q
3k=7a
Cuspitate.
9p=2q
k=a
5p=8q
k=a
9p=10q
k=a
Prolate.
9p=2q
14k=3a
5p=8q
7k=3a
9p=10q
4k=a
Foliate.
9p=2q
k=0
5p=8q
k=0
9p=10q
k=0

PLATE II.

MATHEMATICAL PROBLEMS IN RELATION TO THE HISTORY OF ECONOMICS AND COMMERCE.*

By DAVID EUGENE SMITH, Columbia University.

If students of the history of economics and commerce wish to find a new and interesting field for exploration, and one which is certain to yield results that are worth the labor of cultivation, they will do well to consider the history of problems in arithmetic and algebra as set forth in the manuscripts and early printed books that have come down to us. No doubt some of this field has already been explored, but it is quite certain that only a small portion has thus far come under cultivation. The manuscripts on arithmetic from the thirteenth century to the beginning of printing, the large number of books printed before problems began to represent past as well as contemporary conditions, and the more original text-books of later periods contain a considerable amount of material on the history of commerce and economics that no one seems yet to have studied with any thoroughness.

Considered more broadly, a very good history of civilization could be written from the wide range of problems of mathematics. The emerging of humanity from the stage of mysticism, the development of the science of astronomy, the comprehension of the laws of mechanics and of physics in general, the transition from the agricultural to the industrial stage of a nation, the development of military science, and the rise of commerce are a few of the chapters that might be based upon the problems in mathematics which are easily accessible.

Returning, however, to commercial and economic history, a few examples will be cited to give some notion of the material at the disposal of the student. The problems in the manuscripts and early printed books on arithmetic in the fifteenth century tell us that Venice was then the center of the silk trade, although Bologna, Genoa, and Florence were prominent. Florence was the chief Italian city engaged in the dyeing of cloth. "Nostra magnifica Città di Venetia," as Tartaglia so affectionately and appropriately called her, carried on her chief trade with Lyons, London, Antwerp, Paris, Bruges, Barcelona, Montpellier, and the Hansa towns, besides the cities of Italy. Chiarini (Florence, 1481) indicates the following as the most important cities with which Florence had extensive trade, his spelling being here preserved: Alessandria degypto, Marsilia, Mompolieri, Lisbona, Parigi, Bruggia, Barzalona, Londra, Gostatinopoli, and Dommasco, with the countries of Tunizi, Cypri, and Candia. Tartaglia gives Barcelona, Paris, and Bruges as the leading cities connected with Genoa in trade a half century later.

We also know, from Chiarini's commercial arithmetic (1481), the most important commodities of Florentine trade in the decade before America was discovered. These were rame (brass), stoppa (tow), zoplhi (sulphur), smeriglio

*Extracts from a paper read at an informal session held on the evening between the meetings of the American Mathematical Society and the Colloquium at Cambridge, Mass., September 5, 1917.

(emery), lana (wool), ghalla (gall), trementina (turpentine), sapone (soap), risi (rice), zucchari (sugar), cannella (cinnamon), piombo (lead), lini (flax), pece (pitch), acciai (thread), canapa (hemp), incenso (incense), indachi (indigo), mace (mace), cubeba (cubebs), borage (borax), and the ever-present saffron, the "king of plants," then everywhere used as a *sine qua non* in daily life, and now almost forgotten.

The problems also tell us the cost of the luxuries and the necessities of life. Spanish linen was worth, for example, 94 to 120 ducats per hundred-weight, while Italian linen ran as high as 355 ducats and Saloniki linen as high as 380 ducats. French linen was much cheaper than the latter, selling for 140 ducats. The arithmetics tell us that the linen was baled and sent from Venice to towns like Brescia on muleback.

The problems "delle pigione" tell us that the houses of the bourgeoisie rented in Siena, in 1540, at about 25 to 30 lire per year, while a century later they rented in Florence for from 120 to 300 lire. We also have the prices of sugar, ginger, pepper, and other commodities, showing that these three, for example, were only within the reach of the wealthy.

Hotel life in a grand establishment is also revealed in various problems, of which this one, printed in 1561, is a fair type: "Item / Wenn in einem Gasthause weren 8 Kamern / in jglicher Kamer stünden 12 Bette / in jglichem Bette legen 3 Geste / vnd ein jglicher Gast gebe dem Hausgefinde 6 ß trinckgelt / Wie viel thuts in einer Summa?" The conditions are not at all exaggerated, as many travelers in remote parts of the world to-day can testify.

The early printed arithmetics also show interesting changes in commercial customs and a general rise in the standards of business integrity. For example, the chapter on Die Regel Fusci, very common in the sixteenth century, and relating in part to the adulteration of foods and drugs, would hardly be acceptable to-day either in school or in trade.

The arithmetics also tell very completely the story of the transition from the era of barter to that of the sale of goods for a monetary equivalent. The chapters on "Il baratto," "Stichrechnung," "Troquer," and "Manghelinghe" are among the most interesting ones to be found in these books. They tell us of the influence of the great fairs, they give us lists of these commercial centers, they reveal the relative values of the various commodities in general use, they tell us of the custom of barter in futures (the forerunner of our dealing in futures to-day), and incidentally they explain why a truckman is so called in our time.

An extensive and interesting history of exchange could be written from the problems of arithmetic, including the "cambio menuto, ouer commune," the "cambio reale," the interesting "cambio secco" ("change sec," or "trockener Wechsel") which "non ha humore, ne foglie ne frutto," as Sarava wrote in 1561, and the "cambio fittitio." To these various forms we can trace our standard systems of to-day, and in the study of the "cambio secco" we can understand the law of Henry VII which says that "Eny bargayne . . . by the name of drye exchange . . . be utterly voide."

The transition from partnership in its various forms to the corporations of to-day may well be studied in the problems of the commercial arithmetic, and there may also be followed the genesis, partial decay, and present status of equation of payments. Profit and loss, to-day the most vital topic of business arithmetic, has a long and varied history, and the economics of the problem may be studied in the older books, free from all the modern features of overhead, cost of doing business, and profit on the selling price.

Not only to the economist and the student of commerce is the field a rich one, but it is well worth the study of anyone who may be possessed of doubt as to the relation of mathematics to the daily life of the race. Not only can the history of the problem easily be made the history of commerce and economics, but the history of mathematics can easily be made the history of civilization.

ORGANIZATION OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION OF THE ASSOCIATION.

As a result of preliminary correspondence, a group of Maryland mathematicians held a meeting in New York at the time of the December meeting of the Association and presented a petition to the council for authority to organize a section of the Association in Maryland, Virginia, and the District of Columbia. Professor Abraham Cohen was designated as temporary secretary.

The authority being granted by the council, arrangements were completed for a meeting to organize the Section at Johns Hopkins University on March 3, 1917. Among the 38 persons present at this meeting were the following members of the Association:

O. S. Adams, U. S. Coast and Geodetic Survey, Washington, D. C.
Clara L. Bacon, Goucher College, Baltimore, Md.
Harry Bateman, Johns Hopkins University, Baltimore, Md.
Lillian O. Brown, Hood College, Frederick, Md.
J. A. Bullard, U. S. Naval Academy, Annapolis, Md.
A. B. Coble, Johns Hopkins University, Baltimore, Md.
A. Cohen, Johns Hopkins University, Baltimore, Md.
H. A. Converse, Baltimore Polytechnic Inst., Baltimore, Md.
J. B. Eppes, U. S. Naval Academy, Annapolis, Md.
Angelo Hall, U. S. Naval Academy, Annapolis, Md.
W. M. Hamilton, U. S. Nautical Almanac Office, Washington, D. C.
R. A. Harris, U. S. Coast and Geodetic Survey, Washington, D. C.
A. W. Hobbs, Baltimore, Md.
L. S. Hulburt, Johns Hopkins University, Baltimore, Md.
W. D. Lambert, U. S. Coast and Geodetic Survey, Washington, D. C.
A. E. Landry, Catholic University of America, Washington, D. C.
Frank Morley, Johns Hopkins University, Baltimore, Md.
S. F. Norris, Baltimore City College, Baltimore, Md.

R. E. Root, U. S. Naval Academy, Annapolis, Md.
W. F. Shenton, Johns Hopkins University, Baltimore, Md.
E. R. Smith, The Park School, Baltimore, Md.
J. J. Tanzola, U. S. Naval Academy, Annapolis, Md.
H. R. Tolley, U. S. Dept. of Agriculture, Washington, D. C.

After the adoption of a constitution for the Section, the following officers were elected for the coming year: President, Professor ABRAHAM COHEN, of Johns Hopkins University; Secretary-Treasurer, Professor RALPH E. ROOT, U. S. Naval Academy; Member of the Executive Committee, Mr. WALTER D. LAMBERT, U. S. Coast and Geodetic Survey.

The program consisted of two papers, each followed by a discussion. The first paper, by Professor R. E. Root, was on "The aims and possibilities of this local section," and the discussion was led by Professor A. E. Landry, Mr. W. D. Lambert and Professor Frank Morley. Mr. E. R. Smith and Professor A. B. Coble also took part in the discussion. The second paper, by Professor L. S. Hulburt, was on "A college or university course for teachers of secondary mathematics," and the discussion was led by Professor Clara L. Bacon and Mr. E. R. Smith.

After the program there was a social hour at the Hopkins Club, where the members of the section were the guests of the University at a luncheon.

The constitution of the section provides for holding at least two meetings each year, one in the spring and another in the fall, and the executive committee may call additional meetings when desirable.

RALPH E. ROOT,
Secretary.

SECOND ANNUAL MEETING OF THE OHIO SECTION.

The second annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on April 6, 1917, in connection with the meetings of the Ohio College Association, The Ohio Academy of Science, The Ohio Society of College Teachers of Education, and The Association of Ohio Teachers of Mathematics and Science. Chairman Focke occupied the chair, being relieved by Professor R. B. Allen for an interval.

The following thirty-four persons were registered, all but the last six being members of the Association:

R. B. Allen, Kenyon College; F. Anderegg, Oberlin College; W. E. Anderson, Wittenberg College; G. N. Armstrong, Ohio Wesleyan University; C. L. Arnold, Ohio State University; C. B. Austin, Ohio Wesleyan University; Grace M. Bareis, Ohio State University; Mrs. W. E. Beckwith, Western Reserve University; R. D. Bohannan, Ohio State University; Louis Brand, University of Cincinnati; A. G. Caris, Defiance College; O. L. Dustheimer, Baldwin-Wallace College; Theo. M. Focke, Case School of Applied Science; Harriet E. Glazier, Western College for Women; M. E. Graber, Heidelberg University; Harris

Hancock, University of Cincinnati; E. J. Hirschler, Bluffton College; H. W. Kuhn, Ohio State University; C. C. Morris, Ohio State University; Anna H. Palmie, Western Reserve University; S. E. Rasor, Ohio State University; Hortense Rickard, Ohio State University; Mary E. Sinclair, Oberlin College; S. A. Singer, Capital University; K. D. Swartzel, Ohio State University; C. J. West, Ohio State University; Forbes B. Wiley, Denison University; D. T. Wilson, Case School of Applied Science.

Non-members: P. Biefeld, Denison University; Jacob Bowers, Columbus Trades School; L. D. Parker, Cedarville College; Anna B. Peckham, Denison University; T. Elmer Trott, Mt. Union College; R. B. Wildermuth, Capital University.

The following program was carried out as arranged:

Friday Afternoon.

1. Chairman's Address. "A geometrical presentation of Taylor's series." Professor T. M. FOCKE, Case School of Applied Science.
2. "The application of mathematics to the biological and social sciences." Professor C. J. WEST, Ohio State University.
3. "The actual and apparent path of comet *b* 1916 (Wolf), and the actual path of Mellish's comet (1917*a*)." Professor PAUL BIEFELD, Denison University (Introduced by the Chairman).
4. "The application of Fourier's series to an isoperimetric problem." Professor E. J. HIRSCHLER, Bluffton College.
5. "Formal seminvariants of binary forms." Professor C. C. MORRIS, Ohio State University.
6. "Senior year mathematics for engineering students." Professor LOUIS BRAND, University of Cincinnati.

Friday Evening.

An informal meeting in the nature of a round table was held at the Ohio Union. The following subjects were proposed for discussion:

1. Should there be any absolute requirements in mathematics for the A.B. degree beyond one year of high school algebra and one of plane geometry?
2. How to discover and encourage real mathematical ability.
3. Continuation of the discussion of the afternoon papers.

Each of the first three papers in the afternoon program was illustrated by a number of lantern slides. A photograph of those present at the close of the afternoon meeting was taken. The members of the section dined together at the Ohio Union on Friday evening. Many remained and participated in the meetings of the Association of Ohio Teachers of Mathematics and Science on Saturday.

At the business meeting of the Section, the following officers were elected for the ensuing year:

Chairman, Professor Forbes B. Wiley, Denison University.

Secretary-Treasurer, Professor G. N. Armstrong, Ohio Wesleyan University.

Third member of the Executive Committee, Professor C. L. Arnold, Ohio State University.

Professor Morris emphasized the opportunity and the responsibility of the Ohio Section in connection with the meetings of the American Mathematical Society and the Mathematical Association of America to be held in Cleveland, September 4th to 7th, 1917, and of the Central Association of Science and Mathematics Teachers at Columbus, November 30th to December 1st, 1917.

Abstracts of Papers and Discussions.

1. In his paper on "A geometrical presentation of Taylor's series," Professor Focke suggests that the order of contact of curves be studied before taking up Taylor's theorem. It is then proposed to lead up to the Maclaurin expansion by finding the equation of a curve of the form

$$y = A + Bx + Cx^2 + Dx^3 + \cdots + Rx^n,$$

which shall have a contact of the nth order with $y = f(x)$ at the point where $x = 0$. Then, since the values of A, B, C, etc., turn out to be independent of n, the order of contact of the two curves can be made as high as we choose, provided $f(x)$ and whichever of its derivatives are needed exist when $x = 0$. The necessity of determining the interval of convergency when the number of terms increases without limit was shown by diagrams.

2. Professor West noted that in statistical work which is primarily a matter of accurate counting there is little need for more than the simplest processes of arithmetic. The attempt to draw conclusions true for a whole population, when only a limited number of individuals can be observed, gives rise to a multiplicity of involved problems which cannot be resolved without the aid of highly developed methods. Have real wages increased during the last twenty years; that is, will the laborer's pay purchase more now than at that time? Has the yield of a certain strain of corn been increased? These are typical statistical problems of the more recent kind. In this paper the development of statistical methods is traced and the mathematical problems which arise are outlined. The opportunities which the theory of mathematical statistics offers as a field for mathematical research is especially emphasized.

3. Professor Biefeld presented the report on comets not as a contribution in itself, but rather to suggest an interesting problem for the teacher in practical astronomy or celestial mechanics to be solved by students in his classes in connection with the study of orbits. Namely; the elements of a newly discovered comet being given, to construct:

1. A diagram of the actual path of the comet about the sun to scale. For this it is necessary to compute the true anomaly and radii-vectores for a series of dates.

2. To compute the ephemeris giving the true right ascensions and declinations

and the geocentric distances of the comet, and from the former to plot the apparent path among the fixed stars. It affords an interesting exercise, then, to watch the movement of an object in the sky, find its positions, and compare with computations.

As there are from 2 to 10 telescopic comets every year, new and interesting variations of the problem are brought out. The solution of the problem was illustrated by slides in relation to the above comets.

4. Professor Hirschler outlined a proof for the theorem that a circle encloses a larger area than any other simply closed regular curve of the same length. The curve is represented in parametric form with the length of arc measured from a fixed point as parameter. With the help of a number of recent theorems on Fourier's series its area is calculated in terms of the coefficients of the Fourier expansions for $x = \varphi(s)$ and $y = \psi(s)$. This area is compared with the area of a circle of length l found by integrating the equation $x'^2 + y'^2 = 1$ in terms of the same coefficients.

5. In his paper on "Formal seminvariants of binary forms," Professor Morris showed how the theory of numbers had been applied to the theory of algebraic invariants and seminvariants. He exhibited several fundamental systems for the cubic for different moduli and spoke of the progress that has been made on the quartic.

6. Three types of courses for senior engineering students were considered by Professor Brand:

1. Courses dealing with some phase of engineering with a well developed mathematical theory.

2. Courses presenting a branch of applied mathematics serviceable in engineering practice.

3. Courses in pure mathematics chosen to introduce important concepts not dealt with in elementary courses. As representative of the first type, a course given at the University of Cincinnati on the analysis of statically indeterminate structures was described in some detail. After urging the claims of vector analysis under the second class, the last was exemplified by a course on the number system of algebra, the material being outlined and its advantages discussed.

The evening meeting, held in the large committee room of the Ohio Union, was attended by about twenty-five persons. The discussion of the first topic was so animated and so interesting that the others were not reached and adjournment was forced by necessity of closing the building. Professors Caris, Allen, Bohannan, Morris, Wiley, West, Graber, Glazier, and Anderegg participated in the discussion, both sides of the question being represented. The fact that authority and tradition no longer shield the subjects of the educational curriculum from questioning and attack was emphasized. The claims and counterclaims as to the value of the study of mathematics found adequate presentation, as did also

those of the more recently introduced subjects of study. While strong faith in the validity of the claims of mathematics was apparent, yet there was evident a deep conviction that the teacher could justify himself and his subject only by intelligent, sympathetic, and earnest teaching.

G. N. ARMSTRONG,
Secretary.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Functions of a Complex Variable. By E. J. TOWNSEND, Professor and Head of the Department of Mathematics in the University of Illinois. Henry Holt and Company, New York, 1915. vii + 384 pages.

The present volume is one of the American Mathematical Series of which several have already appeared and of which Professor Townsend is the editor-in-chief. There are but few books in the English language that treat primarily of functions of a complex variable, and when we seek those of an elementary nature which are suitable for an introductory course, to be used by students who have had no further mathematical training than what is generally given in a first course in calculus, the number is very limited. There are the well-known texts, "*Introduction to Analytic Functions*" by Harkness and Morley and "*Theory of Functions of a Complex Variable*" by Burkhardt, translated by Rasor. A book as carefully and well written as the present volume, and which deals with such a fundamental and important field in mathematics, will undoubtedly be well received by both teachers and students.

In Chapter I there is a brief discussion of rational and irrational numbers; then follow the introduction of the complex number and the graphical representation of the same. The fundamental operations of addition, subtraction, multiplication, division, raising to powers, and the extraction of roots are explained for complex numbers, both analytically and graphically. The limitations of the graphic method for extraction of roots are illustrated by means of an example.

The beginning of Chapter II is devoted to the definitions and classifications of functions. The notions and properties of limits of sequences of real numbers are briefly stated, and the ideas are then extended to the realm of complex numbers. In article 13 several fundamental theorems relating to ordinary continuity of $f(z)$ with respect to z, uniform continuity in a region S, and uniform convergence along an arc are stated and proved.

In Chapter III differentiation and integration are taken up. Line integrals and their more general properties are discussed. The path of integration is defined as a curve having the property that it is monotone by segments of a finite number. This lends clearness to the discussions that follow, and obviates difficulties that would arise if a more general curve were chosen. Green's theorem for functions of two real variables, Cauchy's theorem, Cauchy's integral

formula, and the Cauchy-Riemann differential equations are then introduced. The chapter ends with a discussion of Laplace's differential equation and some of its applications to problems in mathematical physics.

Chapter IV is one of the longest in the book and deals with the mapping of given configurations from the Z-plane upon the W-plane and conversely. Detailed discussions are given for the simpler functions such as z^n, e^z, $\log z$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$. Functions of the above form when the argument is real are familiar to the student, but now much new information is added when the variable is complex.

In Chapter V the linear fractional transformations are taken up, beginning with the simpler forms and leading up to the general case. It is shown that the general linear fractional transformation has the properties belonging to a group. The transformations of the plane into itself due to the linear transformations are regarded as a problem in kinematics and the resulting motions are classified as parabolic, hyperbolic, elliptic, and loxodromic.

Chapter VI deals with infinite series. The more important properties of such series with complex terms are considered. The laws of operations with series are explained and illustrated with problems worked out in the text. Double series, uniform convergence, differentiation and integration of series, and power series are taken up in succession.

Chapter VII takes up the discussion of single-valued functions. It begins with a rather thorough treatment of analytic continuation. By the results thus obtained the author formulates more exactly his definition of an analytic function. Following this we have the treatment of singular points and zero points, Laurent's expansion, residues, the fundamental theorem of algebra, rational functions, transcendental functions, Mittag-Leffler's theorem, infinite products and a rather brief discussion of periodic functions.

Chapter VIII, the last in the book, treats of multiple-valued functions. Here the author points out clearly the distinction between multiple-valued analytic functions and multiple-valued expressions representing more than one single-valued analytic function. It is next shown that branch-points are characteristic of the former class. The advantage of the Riemann surface in representing multiple-valued functions is then brought out, and the surface is applied to some typical cases. After discussing some general properties of Riemann surfaces, interesting physical applications to the potential are given. Finally, in Art. 69, the general case of the algebraic function is discussed.

At the end of each chapter there is a list of well-graded exercises to be worked out by the student. It is only by the working out of a large number of such problems in connection with the theory that a student can hope to get a clear idea of the subject. Very excellent geometrical figures are given throughout the text. The typography and general appearance of the book are good. There are but few errors in printing, and those that exist should be of no serious trouble even to a beginner.

HANS H. DALAKER.

UNIVERSITY OF MINNESOTA,

College Algebra with Applications. By E. J. WILCZYNSKI. Edited by H. E. SLAUGHT. Allyn and Bacon, New York, 1916. xx + 507 pages.

"The material included in this book," says the author in his suggestions to the instructor, "probably contains everything ever given under the title *College Algebra* in any American college." The problem of making a selection that will suit the needs of all teachers of algebra is one that might well daunt the most ingenious maker of textbooks. Professor Wilczynski has wisely refrained from undertaking a solution of this problem and has instead given a great storehouse of materials from which the teacher can select to suit his own needs. To the reviewer this seems a very reasonable thing to do. Nothing is so exasperating as the constant worry of some earnest souls about the exact value of this or that tiny theorem in a course. Shall he lay stress on this problem? Shall he insist on a thorough mastery of this principle in the tenth week of the course? Is this the best sequence in presenting the subject? After many years of arranging courses and of trying to find out what material is most available for students the reviewer has given the problem up, as having too many indeterminates, and not enough exact equations. The best he has been able to do is to try to standardize his methods a little, and even in this he has found that one plan of attack will succeed very well with one class and fail with another. It is a problem quite analogous to that of the physician who has first to determine as best he can what is the matter with his patient, and then has the further problem as to the exact drug to use to get the right reaction from the particular patient involved.

The book contains sixteen chapters, beginning with one on the number system that will do much to clear up the difficulties which students (and teachers too) find with the subject of irrationals and complex numbers. Following this come chapters on linear functions, quadratic functions, functions of higher degrees, fractional functions, irrational functions and power functions. Throughout appear many applications to such subjects as the measurement of length, time, mass, the theory of the vernier, slide rule, logarithmic paper, velocity, acceleration, mass, density, and much else of great value to the teacher who is looking for "vitalizing material" for the subject. The author has undertaken to discuss these applications "as carefully as if the book were intended as a treatise on chemistry or physics." There is a chapter on determinants of the first, second and third orders, and another on determinants of higher orders separated from the first by a chapter on permutations and one on probabilities.

Chapter XIII, on simultaneous quadratics, is very full and clear and will be found one of the most helpful in the book. The last three chapters are devoted to the subject of limits, series, both finite and infinite, with a careful statement of the usual tests for convergence. An appendix contains a short table of logarithms and a mortality table.

The reviewer would be slow to predict an immense demand for Professor Wilczynski's book. He feels almost certain that it will not receive half the use which it deserves, the reason being that the average teacher does not want to select. He likes to have an exact statement of just what he is to teach, and the

student likes to have an exact statement of just what he is to learn. This is unfortunate, and not altogether the fault of the teacher. The teacher who omits a chapter from a book is apt to be looked upon with suspicion as a *franc-tireur*, or it may be darkly hinted that he does not understand the chapter well enough to teach it. If it should turn out that there is a large demand for this book the reviewer will take it as a sign that the teacher of mathematics is learning to be less dependent on his textbook. In any event the book will be found in the library of every progressive teacher of algebra.

D. N. LEHMER.

UNIVERSITY OF CALIFORNIA.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

481. Proposed by ROGER E. MOORE, University of Wisconsin.

Show that if the coefficients in the binomial expansion of $(a - b)^n$ (n being a positive integer) be multiplied each to each by the corresponding terms of an arithmetical progression of $(n + 1)$ terms, then the algebraic sum of the $(n + 1)$ products will be zero.

482. Proposed by C. F. GUMMER, Kingston, Ontario.

Find the necessary and sufficient condition that the infinite sequences of positive quantities $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ may be such that the series $a_1x_1 + a_2x_2 + \cdots$ and $b_1x_1 + b_2x_2 + \cdots$ either both converge or both diverge when the x's are any positive quantities.

GEOMETRY.

510. Corrected statement (given incorrectly in the March issue).

Show how to find the equation of a line perpendicular to a side of the triangle of reference and passing through a given point, in a system of homogeneous coördinates, using the condition that two lines be parallel in this system but not the condition that two lines be perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of an escribed circle of the triangle.

514. Proposed by VINCENTE MILLS, Manila, P. I.

Given an equilateral triangle, the length of the sides being unknown, and a point within, the distances from which to the vertices are given, required the length of a side of the triangle and the angles subtended at the given point by the sides of the triangle.

515. Proposed by C. F. GUMMER, Kingston, Ontario.

Show how to cut up a square carpet and make it into three equal square carpets. Estimate the total length of seam in comparison with a side of the original carpet.

CALCULUS.

429. Proposed by N. P. PANDYA, Sojitra, India.

Trace the curve given by the solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{dy}{dx} + \frac{1 - 4x^2}{(1 - x^2)^{3/2}}.$$

430. Proposed by G. PAASWELL, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z axis and pass a plane through the x (or y) axis. Find the area of this surface intercepted by this plane and the xz (or yz) plane.

MECHANICS.

346. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Half the length of one of the equal parts of a uniform heavy string resting in equilibrium over a smooth horizontal indefinitely thin peg is cut off; determine the instantaneous change in the pressure on the peg.

347. Proposed by E. B. ESCOTT, Kansas City, Missouri.

A cord $ABCD$ is suspended from points A and D, which are 20 feet apart in horizontal distance. D is 4 feet lower than A. At B and C are suspended weights of 100 and 200 lbs. $AB = 8$ feet, $BC = 10$ feet, and $CD = 12$ feet. Find angles α, β, γ made by AB, BC, and CD, respectively, with the horizontal. Also find tensions T_1, T_2, and T_3 in AB, BC, and CD.

NUMBER THEORY.

265. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^4 - ax^2 + bx + c = 0$ are rational, prove that $4(a + yz) - 3(y + z)$ is a perfect square, y and z being any two roots of the equation.

245. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that $x^2 + y^2 = (a_1a_2\cdots a_n)^n$ has $4(n + 1)^m$ solutions in integers, in 2^{m+2} of which x and y are relatively prime, the a's being primes of the form $4k + 1$ and n a positive integer.

Note.—The proposer of this problem has changed it to read as above instead of the statement as previously published in the May, 1916, issue.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

469. Proposed by T. H. GRONWALL, New York City.

Show that the equation

$$f(x) \equiv 2ax^4 + (1 - b)x^3 + b(1 - b)x - 2ab = 0,$$

where $0 < b < 1$, $a > 0$ and $a^2 > b$, has only one positive root and that it lies between the roots of

$$g(x) \equiv x^2 - 2ax + b = 0.$$

SOLUTIONS BY D. R. CURTISS, Northwestern University.

I. By Descartes' Rule of Signs, $f(x)$ has only one positive root. Let x_1 be the smaller of the two roots of $g(x) = 0$, and x_2 be the larger. We shall have solved the problem if we establish the inequalities

$$f(x_1) < 0, \qquad f(x_2) > 0. \tag{1}$$

Let us now give b a constant value and let a vary from $\sqrt{b}$ to ∞. If X is either x_1 or x_2 we shall have $g(X) = X^2 - 2aX + b = 0$, and by differentiating with respect to a we obtain

$$\frac{dX}{da} = \frac{X}{X - a}. \tag{2}$$

We now have

$$\frac{df(X)}{da} = f'(X)\frac{dX}{da} + 2(X^4 - b) = \frac{f'(X)\cdot X' + 2(X - a)(X^4 - b)}{X - a}$$

or, in expanded form,

$$\frac{df(X)}{da} = \frac{2X^5 + 6aX^4 + 3(1-b)X^3 + b(1-b)X + 2ab - 2bX}{X - a}. \tag{3}$$

The roots of $g(x) = 0$ satisfy the relations $0 < x_1 < a < x_2 < 2a$, from which we can show that the numerator on the right side of (3) is positive for both values of X. In fact, all the terms are positive except the last; but if $X = x_1 < a$, we have $2ab - 2bX > 0$, so that the numerator is positive in this case, while if $X = x_2$ we have, since $b < a^2$, and $a < x_2 < 2a$,

$$\begin{aligned} 2X^5 + b(1-b)X - 2bX + 2ab &= (2X^5 - b^2X) + (2ab - bX), \\ &> (2a^4 - a^4)X + b(2a - 2a) > 0. \end{aligned}$$

Since $x_1 - a$ is negative and $x_2 - a$ is positive, we thus have

$$\frac{df(x_1)}{da} < 0, \qquad \frac{df(x_2)}{da} > 0.$$

It follows that $f(x_1)$ has its greatest value, and $f(x_2)$ its least value, when $a = \sqrt{b}$, in which case $x_1 = x_2 = a$, and $f(x_1) = f(x_2) = 0$. The relations (1) follow at once.

II. We will show that the positive root of $f(x) = 0$ (there can be only one, by Descartes' Rule) lies between $x = \sqrt{b}$ and $x = a$. The interval $[\sqrt{b}, a]$ is, however, wholly comprised between the two roots of $g(x) = 0$, since $g(0)$ and $g(\infty)$ are positive, while

$$g(\sqrt{b}) = 2b - 2a\sqrt{b} = 2\sqrt{b}(\sqrt{b} - a)$$

is negative, and $g(a) = b - a^2$ is also negative.

We now prove that $f(\sqrt{b}) < 0$ and $f(a) > 0$. In the first place,

$$f(\sqrt{b}) = 2ab^2 + 2b\sqrt{b}(1-b) - 2ab.$$

Since $\sqrt{b} < a$, the substitution of a for $\sqrt{b}$ in the second term above gives

$$f\sqrt{b} < 2ab^2 + 2ab(1-b) - 2ab = 0.$$

On the other hand, we have

$$f(a) = 2a^5 + a^3(1-b) + ab(1-b) - 2ab = 2a^5 + a^3 - ab(a^2 + 1 + b).$$

If we replace each b above by a^2 we obtain the inequality

$$f(a) > 2a^5 + a^3 - a^3(a^2 + 1 + a^2) = 0.$$

Also solved by E. J. MOULTON, J. E. ROWE, H. H. CONWELL, J. A. BULLARD, J. L. RILEY, HORACE OLSON, J. W. BALDWIN, and the PROPOSER.

470. Proposed by ERNEST W. BROWN, Yale University.

There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, where s is an integral multiple of $\frac{1}{2}$.

SOLUTION BY C. F. Gummer, Kingston, Ontario.

Consider the more general problem where the n numbers are chosen at random from the intervals $(a_1, b_1), \cdots (a_n, b_n)$, and the sum is to lie in the interval (a, b).

Let $c_1 = b_1 - a_1, \cdots, c_n = b_n - a_n$.

Let $f_n(x) \cdot dx$ be the probability that the sum of the first r numbers will lie between x and $x + dx$.

Then $f_1(x) = 1/c_1$ when $a_1 < x < b_1$, and zero in other cases, and

$$f_r(x) = \int_{x-b_r}^{x-a_r} \frac{f_{r-1}(\xi) \cdot d\xi}{c_r}.$$

To evaluate $f_n(x)$, let us use the notations

$$\phi(x) = 1 \text{ for } x > 0, \qquad \phi(x) = -1 \text{ for } x < 0, \qquad \phi_r(x) = x^r\phi(x).$$

The function $\phi(x)$ is discontinuous but integrable. Since $\int_0^\alpha \phi(x)dx = \phi_1(\alpha)$, therefore, $\int_a^\beta \phi(x)dx = \phi_1(\beta) - \phi_1(\alpha)$. In like manner,

$$\int_a^\beta \phi_r(x)dx = \frac{\phi_{r+1}(\beta) - \phi_{r+1}(\alpha)}{r+1}.$$

Now

$$f_1(x) = \frac{1}{2c_1}\{\phi(x - a_1) - \phi(x - b_1)\}.$$

Hence,

$$f_2(x) = \frac{1}{c_2}\int_{x-b_2}^{x-a_2} f_1(\xi)d\xi$$

$$= \frac{1}{2c_1c_2}\{\phi_1(x - a_1 - a_2) - \phi_1(x - a_1 - b_2) - \phi_1(x - a_2 - b_1) + \phi_1(x - b_1 - b_2)\}.$$

We thus obtain by induction

$$f_n(x) = \frac{1}{2c_1 \cdots c_n \lfloor n-1}\Sigma(-1)^k\phi_{n-1}(x - \alpha_1 - \beta_2 - \cdots - \kappa_n),$$

where, in the course of the summation, each of the letters $\alpha, \beta, \cdots, k$ is taken to represent either a or b independently of the others, and k is the number of them standing for b in any term.

The probability that the sum lies between a and b is then

$$(1) \qquad \int_a^b f_n(x)dx = \frac{1}{2c_1 \cdots c_n \lfloor n}\Sigma(-1)^{k'-1}\phi_n(\omega - \alpha_1 - \cdots - \kappa_n),$$

where ω stands for a or b, and if it is b it is counted in k'.

We may modify the formula (1) as follows: If a is greater than $b_1 + b_2 + \cdots + b_n$ ($b > a$, $b_r > a_r$), the probability is zero. But the ϕ_n's now reduce to nth powers. Hence,

$$\frac{1}{2c_1c_2 \cdots c_n \lfloor n}\Sigma(-1)^{k'-1}(\omega - \alpha_1 - \cdots - \kappa_n)^n = 0$$

for all values of a and b beyond certain limits, and is therefore identically zero. On adding this zero quantity to the right-hand side of (1) we cause all the ϕ_n's which are not mere nth powers to cancel, and double the others.

Hence, the probability is

$$\frac{1}{\lfloor n\, c_1c_2 \cdots c_n}\Sigma(-1)^{k'-1}(\omega - \alpha_1 - \cdots - \kappa_n)^n,$$

the summation covering all the cases where $\omega - \alpha_1 - \cdots - k_n$ is positive.

It is easy to deduce for the special case propounded that the probability is

$$\frac{1}{\lfloor n}\left\{\left(s + \frac{n+1}{2}\right)^n - {}_{n+1}C_1\left(s + \frac{n-1}{2}\right)^n + \cdots + (-1)^r{}_{n+1}C_r\left(s + \frac{n+1}{2} - r\right)^n\right\},$$

where r is the next integer below $s + (n+1)/2$, and s may be any real number.

GEOMETRY.

Solutions of 497 were received from J. B. Reynolds, Elijah Swift, R. M. Mathews, and H. R Howard after selections for publication were made.

498. Proposed by FRANK R. MORRIS, Glendale, California.

To trisect an angle ABC, on BA and BC take D and E equidistant from B. Using DE as a diameter draw the semicircle $DFGE$. With the same radius and D and E as centers draw arcs locating the points F and G on this semicircle. Connect F and G with B. Prove that this method trisects a right angle and a straight angle and that it does not trisect an oblique angle.

Solution by Roger A. Johnson, Western Reserve University.

Let $x = \angle EBG = \angle FBD$, $y = \angle GBF$, and $\alpha = \angle EBD$. Then $\angle BDF = 150° - \alpha/2$, and $\angle BGF = 90° - y/2$. Applying the law of sines,

$$\frac{\sin x}{\sin\left(150 - \frac{\alpha}{2}\right)} = \frac{FD}{BF}, \qquad \frac{\sin y}{\sin\left(90° - \frac{y}{2}\right)} = \frac{GF}{BF};$$

whence

$$\frac{\sin x}{\sin y} = \frac{\sin\left(150° - \frac{\alpha}{2}\right)}{\sin\left(90° - \frac{y}{2}\right)}. \tag{1}$$

Now in order that $x = y$, we must have either

$$150° - \frac{\alpha}{2} = 90° - \frac{y}{2}, \tag{a}$$

or

$$150° - \frac{\alpha}{2} = 180° - \left(90° - \frac{y}{2}\right). \tag{b}$$

Letting $y = \alpha/3$, we obtain (a) $\alpha = 180°$, (b) $\alpha = 90°$, as the only values of α for which this construction effects the trisection. Conversely, it is very easily seen directly that either a right angle or a straight angle is actually trisected.

To determine roughly the magnitude of the error for angles in general, we may safely replace $\sin(90° - y/2)$ by $\sin(90° - \alpha/6)$; we have then

$$\frac{\sin x}{\sin y} = \frac{\sin\left(30° + \frac{\alpha}{2}\right)}{\sin\left(90° - \frac{\alpha}{6}\right)}. \tag{2}$$

For small angles, the construction is far from accurate; in fact,

$$\lim_{\alpha=0} \frac{\sin x}{\sin y} = \frac{\sin 30°}{\sin 90°} = \frac{1}{2},$$

so that if α is small, it is divided approximately in *fourths*. We find roughly, when $\alpha = 30°$, 60°, 90°, 120°, 150°, and 180°, $(\sin x)/(\sin y) = .710, .879, 1.000, 1.064, 1.066$, and 1.000, respectively. Again, by setting $x = \alpha/2 - y/2$ in (1) and solving for $y/2$, we find

$$\cot\frac{y}{2} = 2\cot\frac{\alpha}{2} + \sqrt{3};$$

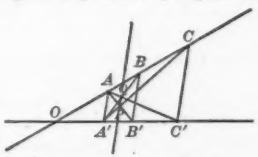
whence we may compute the value of y corresponding to any value of α. For example, if $\alpha = 10°$, $y = 4° 40'$ and $x = 2° 40'$; if $\alpha = 60°$, $y = 21° 46'$ and $x = 19° 7'$; if $\alpha = 135°$, $y = 42° 41'$ and $x = 46° 10'$.

Also solved by Horace Olson.

500. Proposed by R. T. McGregor, Bangor, California.

OABC, *OA'B'C'* are two straight lines such that *AA'*, *BB'*, *CC'* are parallel. *AB'*, *A'C* meet in *P*; *A'B* and *AC'* meet in *Q*. Show by synthetic projective geometry that *PQ* is parallel to *AA'*. Milne's *Projective Geometry*, Chap. I, Ex. 20.

Solution by Claribel Kendall, University of Colorado.



Consider *A*, *B*, *C* as the first, third and fifth sides, respectively of a hexagon; and *A'*, *B'*, *C'* as the fourth, second, and sixth sides, respectively, of the same hexagon. (12), (45); (34), (61) meet in *P* and *Q*, respectively; hence, (23), (56) must meet in a point collinear with *P* and *Q*.

This follows as a special case of Pascal's Theorem for a hexagon inscribed in a conic.* The conic is in this case degenerate, two straight lines. But (23), (56) are parallel and hence must have an infinitely distant point in common with PQ. Therefore PQ is parallel to AA'.

Also solved by MARJORIE L. BROWN, O. S. ADAMS (two methods), F. E. WOOD, NATHAN ALTSHILLER, HANNAH SUFFIN, and H. H. CONWELL.

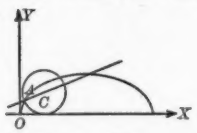
CALCULUS.

416. Proposed by CHARLES N. SCHMALL, New York City.

If A be a point on a cycloid and C the corresponding position of the center of the generating circle, show that AC envelops another cycloid half the size of the first.

SOLUTION BY A. M. HARDING, University of Arkansas.

The coördinates of any point on the cycloid are $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, and the coördinates of the center of the generating circle are given by $x = a\theta$, $y = a$. The equation of AC is



$$\frac{x - a\theta}{\sin\theta} = \frac{y - a}{\cos\theta},$$

or

$$y - a = \cot\theta(x - a\theta).$$

The equation of a cycloid half the size of the given cycloid and having a cusp at O is

$$x = \frac{a\varphi}{2} - \frac{a}{2}\sin\varphi, \qquad y = \frac{a}{2} - \frac{a}{2}\cos\varphi.$$

We propose to show that AC is always tangent to this cycloid.

The equation of any tangent to this cycloid is

$$y - \frac{a}{2}(1 - \cos\varphi) = \cot\frac{\varphi}{2}\left[x - \frac{a}{2}(\varphi - \sin\varphi)\right].$$

Let $\varphi = 2\theta$. This equation then becomes

$$y - a\sin^2\theta = \cot\theta[x - a\theta + a\sin\theta\cos\theta]$$

or

$$y - a = \cot\theta(x - a\theta),$$

which is the same as the equation of AC.

Also solved by C. N. SCHMALL, ELIJAH SWIFT, G. W. HARTWELL, HORACE OLSON, O. S. ADAMS, M. R. GAFFET, R. H. HOWARD, J. B. REYNOLDS, and SHIMPEI NISHIMURA.

417. Proposed by H. S. UHLER, Yale University.

To the degree of approximation indicated show that $(\sqrt{-1})^{\sqrt{-1}} = 0.207879576351$.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

It is not difficult to show, as required in Todhunter's *Plane Trigonometry*, Ed. 1913, pp. 320–21, Examples 266, 275, "that $(a + bi)^{\alpha+\beta i}$ will be wholly real or imaginary if

$$(\beta/2)\log(a^2 + b^2) + \alpha\tan^{-1}(b/a)$$

is (I) zero, or an even multiple of $\pi/2$; or (II) an odd multiple $\pi/2$". In the problem, $a = \alpha = 0$, and $b = \beta = 1$, the conditions corresponding to (I), 0 being called an even number. $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{1}{2}\pi}$, and the numerical value can be tested by using enough decimals in the values of e and π.

* Cremona, *Elements of Projective Geometry*, translated by Leudensdorf, Articles 88 and 153, 3d edition.

The problem is of some historical interest, being due to John Bernoulli. The distinguished American astronomer, G. W. Hill, proposed it in *The Analyst*, Vol. II, January, 1875, p. 31, Ex. 56, with no reference to its history. It was solved by Walter Siverly, and Henry Heaton in the succeeding number of *The Analyst*, by the following method:

By Euler's Theorem, $\cos\theta + i\sin\theta = e^{i\theta}$, where θ may be $(4n+1)(\pi/2)$. Assume $n = 0$. Then $i = e^{(\pi/2)i}$ and $(i)^i = e^{-\frac{1}{2}\pi} = 0.2078795763507$.

Note.—By an oversight, this problem was placed under Calculus.

Solutions were also received from J. B. Reynolds, Paul Capron, O. S. Adams, and E. B. Escott.

Mr. Escott, using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .20787957634917907781.—Editors.

MECHANICS.

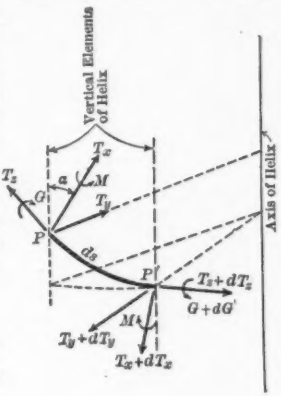
300. Proposed by V. M. SPUNAR, Chicago, Ill.

A helical spring is composed of twenty turns of steel wire 0.258″ in diameter, the diameter of the coil being 3″. If the spring is compressed by a force of 50 lbs., what is the maximum stress in the coil, its axial compression and its resilence?

Solution by G. Paaswell, N. Y. City.

This solution follows the method given by Love in his treatise on Elasticity.

At any point, P, of the helix, take a system of rectilinear coördinates chosen as follows: the tangent, as the z axis; the radial element as the y axis; and the normal to the two above, as the x axis. The motion of the curve may be analyzed into a curvature k about the x axis described by the z axis and a twist, t, about the z axis described by the x axis. These are, respectively, if the pitch of the helix is a, $\cos^2 a/r$, $(\sin a \cos a)/r$, where r is the radius of the coil. This curvature, k, causes a bending moment, M, and the twist, t, causes a torsional moment, G.



If s is the total length of the helix; n, the number of turns and h the height of the coil; then $h = s\sin a$ and $s\cos a = 2\pi nr$.

Assume the spring subjected to an elongating force R and consider a small free portion of length ds (see figure). The twist from P to P' is tds and the curvature, kds. From ordinary statics,

$$\Sigma_x = \Sigma_y = \Sigma_z = 0,$$

whence

$$T_x - (T_x + dT_x) + tds(T_y + dT_y) = 0, \quad (1)$$

$$T_y - (T_y + dT_y) - tds(T_x + dT_x) + kds(T_z + dT_z) = 0, \quad (2)$$

$$T_z - (T_z + dT_z) - kds(T_y + dT_y) = 0, \quad (3)$$

$$\Sigma_{\text{mom. } x} = \Sigma_{\text{mom. } y} = \Sigma_{\text{mom. } z} = 0,$$

$(T_y + dT_y)ds = 0$; whence $T_y = 0$ and $(T_x + dT_x)ds + Gkds - Mtds = 0$.

From the above, we get at once $T_z - Mt + Gk = 0$ and $tT_x - kT_z = 0$, or $T_x \tan a = T_z$.

The applied loading deforms the helix into a new one with pitch a' and radius r'. The new curvature and twist is, then, t', k'. Assuming that the original curve was under no stress, the bending and torsional moments are given, respectively, by the difference in the curvatures and twists, multiplied by the flexural and torsional rigidities. Then, $M = EI(k' - k)$ and $G = E_s I_s(t' - t)$, where E is Young's Modulus; E_s is the shearing modulus, I is the moment of inertia of the wire about a diameter; and I_s is the polar moment of inertia of the wire. If Poisson's ratio is taken as 1/4, then $E_s = \frac{2}{5}E$. Representing EI by A, then $M = A(k' - k)$ and $G = \frac{1}{5}A(t' - t)$. (6) can now be written $T_z = A\{t'(k' - k) - \frac{1}{5}k'(t' - t)\}$. Since the resultant stress must equal R, $R = T_z \sec a'$. Likewise, the resultant moment is

$$W = M\cos a' + G\sin a'.$$

There is, finally,

$$R = A\left\{\frac{\sin a'}{r'}\left(\frac{\cos^2 a'}{r'} - \frac{\cos^2 a}{r}\right) - \frac{4}{5}\frac{\cos a'}{r'}\left(\frac{\sin a' \cos a'}{r'} - \frac{\sin a \cos a}{r}\right)\right\}$$

and

$$W = A\left\{\cos a'\left(\frac{\cos^2 a'}{r'} - \frac{\cos^2 a}{r}\right) + \frac{4}{5}\sin a'\left(\frac{\sin a' \cos a'}{r'} - \frac{\sin a \cos a}{r}\right)\right\}.$$

The distortion of the helix will change the total angle φ, turned by the helix into $\varphi + d\varphi$; where φ is $2\pi n$; and the height of the helix from h to $h + dh$. Since $\varphi = (s \cos a)/r$ then $\varphi + d\varphi = (s \cos a')/r$ and $h + dh = s \sin a'$. Taking $d\varphi$ and dh as infinitesimal we have $s \cos a' = s \cos a - \tan a dh$. Since

$$\frac{1}{r} = \frac{\varphi}{\sqrt{s^2 - h^2}},$$

then

$$\frac{1}{r^2} = \frac{\varphi + d\varphi}{\sqrt{s^2 - (h + dh)^2}}.$$

After proper reductions and substitutions

$$R = \frac{A}{s^3}\left(\frac{4s^2 + h^2}{5(s^2 - h^2)}\varphi^2 dh - \frac{\varphi h}{5}d\varphi\right) \quad \text{and} \quad W = \frac{A}{S^3}\left(-\frac{\varphi h}{5}dh + \frac{5s^2 - h^2}{5}d\varphi\right).$$

There are two possible cases: the terminal is held fast against twist and hence $d\varphi = 0$; and the terminals are free to turn and the resultant moment W is zero. The former is the usual case in practice and will be the only one discussed here. Either condition, with R given, renders the problem a determinate one. Introducing the former condition, then

$$R = \frac{A}{5s^2}\frac{4s^2 + h^2}{s^2 - h^2}\varphi^2 dh \quad \text{and} \quad W = -\frac{A\varphi h}{5S^3}dh.$$

If p is the radius of the wire and S is the unit shear, we have, since the section is a circle,

$$S = pW/I_s.$$

The resilience must equal the work done, whence Res. $= \frac{1}{2} Rdh$.

With the conditions as given in the problem, dh is first to be found, and noting that $s^2 - h^2 = \varphi r^2$, there is

$$dh = \frac{5s^2 r^2 R}{A(4s^2 + h^2)}.$$

Further, since h^2 is usually infinitesimal in comparison with $4s^2$, the formula may be simply written

$$dh = \frac{5sr^2 R}{4A}.$$

This is the expression used in the numerical substitution below.

Note, that within reasonable limits of the pitch, s is practically constant so that the shortening of the spring is independent of the pitch.

With the data given in the problem, $r = 1.5''$; $p = 0.129''$; $R = 50$ lbs. and E is assumed as 30,000,000 lbs. so that $A = 6{,}570$, $n = 20$. The horizontal projection of the length of the spring is $2\pi nr$, which is equal to $188.5''$, so, that for small pitches, s may be taken about 190. The shortening is then $4.1''$. Substituting this in the expression for W and then introducing this value of W for finding S, the maximum stress, *i. e.*, shear, is 855 lbs. per sq. in. The resilience is found to be 102.5 in. lbs.

329. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A smooth circular table is surrounded by a smooth circular rim. Show that the ball, whose coefficient of restitution is e projected along the table from a point in the rim making an angle $\tan^{-1} e^{3/2}$ with the radius through the point, will return to the point of projection after three rebounds.

I. Solution by G. Paaswell, N. Y. City.

Before attacking this special problem let us establish the general case of k points of contact, *i. e.*, of $k-1$ rebounds. Denote the reciprocal of e by r and tan a by s (referring to the figure). From the laws of elastic impact, tan $b = r$ tan a; then tan $c = r^2$ tan a; tan $d = r^3$ tan a and so forth. From the figure, $2a + 2b + 2c + \cdots = (k-2)\pi$; whence

$$\tan^{-1} s + \tan^{-1} rs + \tan^{-1} r^2 s + \cdots = \frac{\pi}{2}(k-2)$$

It may be shown by developing a few cases and then by induction that

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \tan^{-1} a_3 + \tan^{-1} a_4 + \cdots + \tan^{-1} a_k$$

$$= \tan^{-1}\left(\frac{{}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \cdots \pm {}_k\Sigma_k}{1 - {}_2\Sigma_k + {}_4\Sigma_k \cdots \pm {}_{k-1}\Sigma_k}\right) = \frac{\pi}{2}(k-2),$$

when k is odd and

$$= \tan^{-1}\left(\frac{{}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \cdots \pm {}_{k-1}\Sigma_k}{1 - {}_2\Sigma_k + {}_4\Sigma_k \cdots \pm {}_k\Sigma_k}\right) = \frac{\pi}{2}(k-2),$$

when k is even.

${}_p\Sigma_k$ represents the sum of the k arguments taken p at a time. Taking the tangent of both members $\left(\text{the second member is } (k-2)\frac{\pi}{2}\right)$ and noting that when k is odd, $\tan(k-2)\frac{\pi}{2}$ is infinite and hence the denominator must vanish, and similarly, when k is even, the numerator of the corresponding fraction must vanish, we have

$${}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \pm \cdots\cdots {}_{k-1}\Sigma_k = 0 \ (k \text{ is even}),$$

$$1 - {}_2\Sigma_k + {}_4\Sigma_k \pm \cdots\cdots {}_{k-1}\Sigma_k = 0 \ (k \text{ is odd}).$$

When the sequence of a's forms a geometric progression it may be shown without difficulty that

$${}_p\Sigma_k = r^{p(p-1)/2} s^p \frac{1 - r^{-Ap}}{1 - r}; \qquad A_p = \frac{k!}{(k-p)!\,p!};$$

and the condition equations become

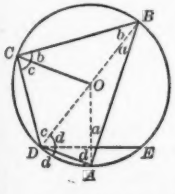
$$s\frac{1 - r^{A_1}}{1 - r} - r^3 s^3 \frac{1 - r^{A_3}}{1 - r} + r^{10} s^5 \frac{1 - r^{A_5}}{1 - r} \cdots \pm r^{(k-1)(k-2)/2} s^{k-1} \frac{1 - r^k}{1 - r} = 0, \ k, \text{ even};$$

$$1 - rs^2 \frac{1 - r^{A_2}}{1 - r} + r^6 s^4 \frac{1 - r^{A_4}}{1 - r} \cdots \pm r^{(k-1)(k-2)/2} s^{k-1} \frac{1 - r^k}{1 - r} = 0, \ k, \text{ odd}.$$

These are the general equations for any number of rebounds. Thus, $k = 2$ (one rebound) $s(1 + r) = 0$, or $s = 0$, $a = 0$; $k = 3$ (two rebounds), $1 - rs^2 \frac{1 - r^3}{1 - r} = 0$, $s^2 = \frac{1}{r(1 + r + r^2)}$

$a = \tan^{-1}\sqrt{\frac{e^3}{1 + e + e^2}}$; $k = 4$ (3 rebounds, the given case), $s\frac{1 - r^4}{1 - r} - r^3 s^3 \frac{1 - r^4}{1 - r} = 0$;

$1 - s^2 r^3 = 0$, $s = \frac{1}{r^{3/2}}$; whence $a = \tan^{-1} e^{3/2}$.



II. Solution by Charles A. Hutchinson, Wittenberg College, Springfield, Ohio.

Let A be the point of projection, and B, C, D, the points of first, second and third rebounds, respectively.

Assume that after the third rebound, the ball does not pass along DA, but along DE.

Then tan $a = \sqrt{e^3}$, tan $b = 1/e \cdot$tan $a = \sqrt{e}$, tan $c = 1/e \cdot$tan $b = 1/\sqrt{e}$, and tan $f = 1/e \cdot$tan $c = 1/\sqrt{e^3} =$ cot a. Hence, $f = 90° - a$.

Since $\tan c = 1/\sqrt{e} = \cot b$, therefore, $c = 90° - b$. Hence, $a + b + c + f = 180$. But $a + b + c + d = 180°$. Hence, $d = f$.

Hence, DA and DE coincide and the ball does return to A.

Also solved by HORACE OLSON, A. M. HARDING, and HAROLD T. DAVIS.

NUMBER THEORY.

230. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Find three numbers such that their sum, the sum of their squares, and the sum of their cubes shall be a cube.

SOLUTION BY E. T. BELL, Seattle, Washington.

Presumably the proposer wished three *integers*, (x, y, z) satisfying the conditions. No restrictions being imposed, we may assume one of the cubes to be zero, and choose $(x + y + z) = 0$, which simplifies the problem very considerably, reducing it in effect to the solution in integers (m, v) of

$$m^2 = 36v^2 + 1. \tag{1}$$

A solution (x, y, z) is:

$$\begin{aligned} x &= 146, & x^2 &= 21\ 316, & x^3 &= 3\ 112\ 136, \\ y &= -1\ 314, & y^2 &= 1\ 726\ 596, & y^3 &= -2\ 268\ 747\ 144, \\ z &= 1\ 168, & z^2 &= 1\ 364\ 224, & z^3 &= 1\ 593\ 413\ 632, \\ w &= -876, & & & w^3 &= -672\ 221\ 376, \end{aligned}$$

whence

$$\begin{aligned} x + y + z &= 0^3, \\ x^2 + y^2 + z^2 &= x^3, \\ x^3 + y^3 + z^3 &= w^3 = (-6x)^3, \end{aligned}$$

which may be most readily checked from the resolutions into prime factors:

$$x = 2 \times 73; \quad y = -2 \times 3^2 \times 73; \quad z = 2^4 \times 73; \quad w = -2^2 \times 3 \times 73;$$

clearly, (xt^3, yt^3, zt^3), where t is an arbitrary integer, is also an integral solution.

More generally, (m, v) being any solution of (1), and k arbitrary,

$$\begin{aligned} x &= 18(m^2 + 3)(m + 1)^2(m - 1)^2k^3, \\ y &= -9(m^2 + 3)(m + 1)^2(m - 1)^2k^3, \\ z &= 9(m^2 + 3)(m + 1)^2(m - 1)^2k^3, \end{aligned} \tag{2}$$

give

$$\begin{aligned} x + y + z &= 0^3, \\ x^2 + y^2 + z^2 &= \{2^3 3^4 k^2 v^4 (m^2 + 3)\}^3 = \{2^3 3^4 v^4 k^3 (3^2 v^2 + 1)\}^3, \\ x^3 + y^3 + z^3 &= \{-2^3 3^7 (m^2 + 3) v^7 k^3\}^3 = \{-2^7 3^7 v^7 k^3 (3^2 v^2 + 1)\}^3, \end{aligned} \tag{3}$$

which, for $k = 2^{-4}3^{-2}$, and the solution $(m, v) = (17, 2)$ of (1) gives the solution first stated.

Note.—The analysis for solution 2, which would take about two pages of the MONTHLY, can be written out if thought of sufficient interest, but it involves nothing new. Incidentally, the above solution carries with it that of many other curious indeterminate systems, when (3) is combined with Newton's formulæ for the sums of like powers of the roots of an algebraic equation; thus, we are shown how to find integers (a, b, c, p, r) satisfying

$$\begin{aligned} a^5 + b^5 + c^5 &= 30p^3, \\ a^7 + b^7 + c^7 &= 28r^3. \end{aligned}$$

There is an infinite chain of such results, which are all more or less in the nature of accidents.

246. Proposed by ALBERT A. BENNETT, Princeton University.

Prove that

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n-\left(\frac{a-\sqrt{b}}{2}\right)^n\right]$$

is an integer for every positive integral value of n, whenever a is an odd integer, positive or negative, and $b \equiv 1 \pmod 4$.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Taking out the factor

$$\frac{a+\sqrt{b}}{2}-\frac{a-\sqrt{b}}{2}=\sqrt{b},$$

we have

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n-\left(\frac{a-\sqrt{b}}{2}\right)^n\right]=\sum_{s=0}^{n-1}\left(\frac{a+\sqrt{b}}{2}\right)^{n-s-1}\left(\frac{a-\sqrt{b}}{2}\right)^s.$$

Let

$$\frac{a+\sqrt{b}}{2}=x, \qquad \frac{a-\sqrt{b}}{2}=y.$$

Then this expression is

(I) $$(x^{n-1}+y^{n-1})+xy(x^{n-3}+y^{n-3})+x^2y^2(x^{n-5}+y^{n-5})+\cdots+L;$$

where $L = x^{(n-1)/2}y^{(n-1)/2}$ if n is odd; and $L \equiv x^{(n-2)/2}y^{(n-2)/2}(x+y)$ if n is even.

Let $a = 2k+1$, $b = 4l+1$, k and l being integers. Then

$$xy=\frac{a^2-b}{4}=k^2+k-l,$$

an integer, so that any positive integral power of xy is an integer.

Also $x+y=a$ and $x^2+y^2=(x+y)^2-2xy=a^2-2(k^2+k-l)$ are integers. Further,

$$x^p+y^p=(x+y)^p-\Sigma\frac{p!}{s!(p-s)!}x^sy^s(x^{p-2s}+y^{p-2s});$$

so that if x^p+y^p is integral for a positive integral value of p, it is integral for a value of p greater by 2; but it is integral for $p=1$ and for $p=2$, and so is integral for any positive integral value of p.

Thus each term of (I) is integral, and I is an integer.

Also solved by ELBERT H. CLARKE.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO INFINITESIMAL METHODS IN GEOMETRY.

BY E. B. WILSON, Massachusetts Institute of Technology.

The old method of using infinitesimals to find by geometric considerations various properties of figures was given a prominent place by W. E. Byerly in his *Differential Calculus*, but has been almost wholly discarded by recent writers. It is a question worthy of discussion whether the present neglect of the method is entirely wise. An example or two may be given.

1. *Consider the problem of the maximum triangle inscribed in a convex curve, or the minimum triangle circumscribed about the curve.*

If ABC is the maximum triangle inscribed in the curve, a displacement of any vertex, say C, along the curve, can only change the area of the triangle by an infinitesimal of higher order than the first, and hence the altitude from C to AB must change only by an infinitesimal of higher order. This means that the tangent to the curve at C must be parallel to the side AB. We have, therefore, the proposition that the maximum triangle is that for which the tangents to the curve at the vertices are parallel to the opposite sides of the triangle. Incidentally, the geometry of the figure shows that the triangle formed by the three tangents has its three sides bisected by the points of tangency.

If the minimum triangle circumscribed about the curve is desired, let PQR be the triangle. The displacement of the point of tangency of one side must change the area of the triangle by an infinitesimal of higher order. A consideration of the small triangles added to and subtracted from PQR by the change of one side to a neighboring point of tangency, shows at once that the triangle PQR has its sides bisected by their points of tangency. Incidentally the geometry of the construction shows that the inscribed triangle formed by joining the three points of tangency has its sides parallel to the sides of PQR.

It is thus seen that the solutions for the maximum inscribed triangle and the minimum cricumscribed triangle are identical in so far as the determination of three points on the curve is concerned. Is there any *easy* way of reaching these results by the exclusively analytical methods now in vogue?

As applied to the circle, it is a matter of the simplest plane geometry to show that the above construction calls for equilateral triangles. Is there any easier way than this for finding the maximum and minimum triangles for the special case of the circle?

Any ellipse may be regarded as the orthogonal projection of a circle. In such a projection areas are multiplied by a constant, the cosine of the angle between the planes. The eccentric angle in the ellipse is identical with the central angle in the circle. Hence we see that the maximum triangle inscribed in the ellipse is any triangle whose vertices are equally spaced along the ellipse when measured by the eccentric angle (or by the sectorial area; *i. e.*, the central radii to the vertices of the triangle divide the ellipse into three sectors of equal area). The minimum triangle about the ellipse needs no special attention, because of the general theorem mentioned above.

2. *Consider the problem of finding the point P in the plane of three points ABC such that the sum of the distances $PA + PB + PC$ shall be a minimum.*

If this problem is solved by the straightforward analytical method, the analysis appears complicated, owing to the three radicals used to express the distances.

The geometric solution is easy. Let P be the point. Displace P along a line perpendicular to one of the lines PA, PB, PC, say PC. Then PC changes by an infinitesimal of the second order, and $PA + PB$ must change by an

infinitesimal of higher order than the first; that is, $PA + PB$ is constant up to and including the first order. Hence (as shown by the laws of reflection for light, or from the properties of the ellipse), PA and PB must make equal angles with PC. By symmetry, the angles subtended at P by the three sides of the triangle ABC must all be 120°.

It is now easy to construct the point P. Draw upon two sides of the triangle arcs of 120°. Their intersection determines P, and the arc of 120° on the third side of the triangle passes through P. (The construction breaks down when one angle of the triangle is greater than 120°, and in this case the vertex of that angle is the solution of the problem.)

I should be glad to see other analytic solutions of the problem which are as simple. See the solution by Dunham Jackson in the January 1917 MONTHLY, which appeared after this note was sent to the editor.

II. Relating to a Problem in Minima Discussed by Professor Dunham Jackson in the January, 1917, Number (p. 46) of this Monthly.

By Roger A. Johnson, Western Reserve University, Cleveland, Ohio.

The problem is to determine a point the sum of whose distances from the vertices of a given triangle shall be a minimum.

Professor Jackson has apparently overlooked the fact that this problem admits an elementary geometric solution due to Steiner (*Collected Works*, Vol. II, p. 729). The following is a development of this solution; for more detailed treatment and historical notes, see Emmerich, *Die Brocard'sche Gebilde*, § 42.

Lemma I. The sum of the distances to the sides of an equilateral triangle, from a point inside the triangle, is constant and equal to the altitude. Moreover, if from any point the perpendiculars to the sides of the equilateral triangle make angles of 120° (and not 60°) with one another, the point is inside the triangle.

Lemma II. Let ABC be a triangle, each angle of which is less than 120°. There is a point P, and only one, such that $\angle BPC = \angle CPA = \angle APB = 120°$.

Lemma III. Through A, B, C, draw MN, NL, LM, perpendicular to PA, PB, PC, respectively. Then LMN is an equilateral triangle and P is inside it.

THEOREM. *If Q is any point of the plane, other than P, then*

$$QA + QB + QC > PA + PB + PC.$$

For $QA + QB + QC$ is greater than the sum of the perpendiculars from Q to MN, NL, LM; and the sum of these perpendiculars is greater than or equal to $PA + PB + PC$, according as Q lies outside or inside triangle LMN.

The point P is one of two points called *isogonic centers*, which have numerous interesting properties (cf. Emmerich, *l. c.*). If any angle of the triangle is greater than 120°, it falls outside the triangle and therefore fails to yield a minimum. As Professor Jackson indicates, it is not hard in this case to show that the vertex of the obtuse angle is the desired point. To do this by the same method we have

just used, we assume that angle B is greater than 120°. Through A and C, respectively, let XAY and XBZ be drawn perpendicular to BA and BC respectively; through B draw a line YZ so that $\triangle XYZ$ is isosceles, and $\angle YXZ$ is less than 60°; then $ZY < XY$.

Let PF, PG, PH, be perpendicular to XY, YZ, ZX, respectively.

Then

$$XY \cdot PF + XZ \cdot PH + YZ \cdot PG = XY \cdot BA + XZ \cdot BC.$$

Let $m = XY = XZ$. Then, since $YZ \cdot PG < XY \cdot PG$,

$$m(PF + PG + PH) > m(BA + BC),$$

and

$$PF + PG + PH > BA + BC.$$

Also, as before, $PA + PB + PC \geqq PF + PG + PH$, which establishes the result.

III. Relating to the Exponential Function.

By Otto Dunkel, Washington University.

An interesting treatment of the elementary transcendental functions was given by A. Hurwitz in the *Mathematische Annalen*, Vol. 70, 1911, entitled "Über die Einführung der elementaren transzendenten Funktionen in der algebraischen Analysis." The method used by Hurwitz was somewhat similar to the one used by Professor Huntington in his article "An Elementary Theory of the Exponential and Logarithmic Functions" in the September, 1916, number of the Monthly, pp. 241–246, except that Hurwitz discussed first the function $\log x$ and then derived sequences for the definition of e^x of the type used by Huntington. About the time of the appearance of Hurwitz's article the writer developed a treatment of the exponential function similar to the one given by Huntington but somewhat simpler in the fact that the inequality $(1 + d)^m > 1 + md$ was not required. This makes the proof more elementary, as the proof of this inequality in Huntington's article is made to depend partly upon the binomial theorem and partly upon an additional proof for a remaining case. This advantage is gained by using sequences in which each exponent is double the preceding, for then it is a very easy matter to prove the increasing and decreasing character of the two sequences, and having done this the sequences themselves supply the place of the above mentioned inequality in the subsequent reasonings. A brief sketch of how this can be effected is given below.

Given the sequences

$$(A)\quad 1+x,\ \left(1+\frac{x}{2}\right)^2,\ \left(1+\frac{x}{4}\right)^4,\ \left(1+\frac{x}{8}\right)^8,\ \cdots,\ \left(1+\frac{x}{m}\right)^m,\ \left(1+\frac{x}{2m}\right)^{2m},\ \cdots$$

$$(B)\quad (1-x)^{-1},\ \left(1-\frac{x}{2}\right)^{-2},\ \left(1-\frac{x}{4}\right)^{-4},\ \left(1-\frac{x}{8}\right)^{-8},\ \cdots,\ \left(1-\frac{x}{m}\right)^{-m},\ \left(1-\frac{x}{2m}\right)^{-2m},\ \cdots,$$

where x is any real number, and considering only the terms for which $m > |x|$, we shall prove the following: (a) Sequence (A) increases; (b) Sequence (B) decreases; (c) Each A is less than the corresponding B and the difference approaches zero as m increases.

We have for any real value of x

$$\left(1+\frac{x}{2m}\right)^2 = 1+\frac{x}{m}+\frac{x^2}{4m^2} > 1+\frac{x}{m},$$

and, since $1 + x/m$ is positive, we have also

$$\left(1+\frac{x}{2m}\right)^{2m} > \left(1+\frac{x}{m}\right)^m.$$

This proves (a), and (b) follows at once from the above inequality by replacing x by $-x$ and reversing the resulting inequality.

To prove (c) we have

$$\left(1-\frac{x}{m}\right)^{-m} - \left(1+\frac{x}{m}\right)^m = \left(1-\frac{x}{m}\right)^{-m}\left[1-\left(1-\frac{x^2}{m^2}\right)^m\right]$$

$$< \left(1-\frac{x}{m}\right)^{-m}\frac{x^2}{m} \quad \text{when } m > x^2.$$

It is easily seen that each of the two factors on the right in the equality is positive and hence the first part of (c) is true. The inequality is seen to be true by replacing x in (A) by $-x^2/m$ and using (a). Since the factor $(1 - x/m)^{-m}$ decreases and x^2/m approaches zero as m increases the second part of (c) is true.

If we call the common limit of the sequences (A) and (B) $\exp x$, we have the important inequalities

$$1+h < \left(1+\frac{h}{m}\right)^m < \exp h < \left(1-\frac{h}{m}\right)^{-m} < (1-h)^{-1}, \quad |h| < 1, \quad m = 2^t,$$

from which follow the remaining properties of the function. For example, since

$$\frac{\left(1+\frac{x}{m}\right)^m\left(1+\frac{y}{m}\right)^m}{\left(1+\frac{x+y}{m}\right)^m} = \left[1+\frac{xy}{m(m+x+y)}\right]^m,$$

if we take m so large that $xy/(m + x + y) = h$ is less than unity in absolute value, the above inequalities show that

$$\frac{\exp x \exp y}{\exp(x+y)} = 1.$$

The derivative of $\exp x$ follows at once from the same inequalities. There is therefore an advantage in beginning the sequences with $1 + x$ and $(1 - x)^{-1}$

respectively. The sequences given here contain, of course, those in Huntington's paper.

IV. Relating to Infinite Products for sin z and cos z.

By M. B. Porter, University of Texas.

As the elementary method for obtaining these products primarily applies only to real variables and involves the evaluation of a somewhat troublesome limit and as such proofs as those of Borel's *Fonctions Entières*, p. 82, require a recondite theorem of Hadamard concerning functions of finite order, it seemed to the writer that the following considerations might interest the readers of the Monthly.

We start with the polynomial

$$f_n(z) = z \prod_{m=-n}^{m=n} \left(1 - \frac{z}{mc}\right), \quad c \text{ real},$$

which evidently approaches the analytic limit

$$F(z) = z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2c^2}\right)$$

uniformly as n becomes infinite.

Evidently $f_n'(z)$ has all of its roots real by Rolle's theorem and since it is an even function these roots are equal in pairs though of opposite sign, and separate the roots of $f_n(z) = 0$; if we denote the ith root by c_i we have

$$(i - 1)c < c_i < ic. \tag{1}$$

Now consider

$$\frac{f_n'(z)}{f_n(z)} = \sum_{-n}^{n} \frac{1}{z - mc} \equiv \sum_{m=k-s}^{m=k+s+1} \frac{1}{z - mc} + R_n(z). \tag{2}$$

As n becomes infinite this is evidently convergent and uniformly so if we cut out the neighborhoods of the point $z = mc$; and since

$$\left| \sum_{-n-r}^{n+r} \frac{1}{z - mc} - \sum_{-n}^{n} \frac{1}{z - mc} \right| < \epsilon,$$

for n large enough, it is evident that the zeros of $f_n'(z) = 0$ are as close as we please to $(k + \frac{1}{2})c$ inside of a circle as large as we please if n is taken large enough; for $(k + \frac{1}{2})c$ is obviously a root of the function defined by the last summation in (2). Now

$$f_n'(z) \equiv \prod_{i=-n}^{i=n} \left(1 - \frac{z}{c_i}\right) = \prod_{i=-m}^{i=m} \left(1 - \frac{z}{c_i}\right) \prod_{i=m+1}^{i=n} \left(1 - \frac{z^2}{c_i^2}\right) (m < n).$$

Comparing the last product on the right with

$$\prod_{m+1}^{n}\left(1-\frac{z^2}{m^2c^2}\right),$$

we see that it is as near unity as we please while the first product is as close to

$$\prod_{-m}^{m}\left(1-\frac{z}{(m+\frac{1}{2})c}\right)$$

as we please if n is large enough.

Thus we see that

$$\operatorname*{Lim}_{n=\infty} f_n'(z) = F'(z) = \operatorname*{Lim}_{n=\infty}\prod_{-n}^{n}\left(1-\frac{z}{(m+\frac{1}{2})c}\right).$$

By the same reasoning we show that

$$F''(z) = -\frac{8}{c^2}\sum_{1}^{\infty}\frac{1}{(2n+1)^2}F(z), \tag{3}$$

and integrating the differential equation (3) we have

$$F(z) = \sin\frac{\sqrt{8\Sigma}}{c}z = kz\prod_{1}^{\infty}\left(1-\frac{z^2}{n^2c^2}\right).$$

The constant k is seen to be $\sqrt{8\Sigma}/c$ and $\sqrt{8\Sigma}$ can be evaluated by noting that since sine x has the period 2π and the product the period $2c$ we have

$$\sqrt{8\Sigma} = \pi.$$

The product for $\cos \pi z/c$ can be gotten in the same way, since

$$F'(z) = \cos\frac{\sqrt{8\Sigma}}{c}z$$

is the solution of (3) required.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Assistant Professor C. B. UPTON has been promoted to an associate professorship of mathematics at Teachers College, Columbia University.

Mr. W. H. WILSON, University of Illinois, has accepted an instructorship in mathematics at the Massachusetts Institute of Technology.

Professor A. M. HARDING has been promoted to a full professorship of mathematics at the University of Arkansas.

Dr. G. R. CLEMENTS, of the University of Wisconsin, has been appointed instructor in mathematics at the U. S. Naval Academy, Annapolis.

Professor E. H. MOORE, University of Chicago, has been appointed chairman of the Mathematical Committee of the National Research Council.

Word has been received announcing the death of Mr. J. A. COLSON, at Searsport, Me., February 13, 1917. Mr. COLSON was a charter member of the Association.

Dr. J. W. NICHOLSON, for forty years professor of mathematics in the University of Louisiana, and author of school and college text-books, died on March 22. Dr. NICHOLSON was a charter member of the Association.

Professor WILLIAM BEEBE, of the mathematical faculty of Yale University, died on March 11, aged sixty-six. He graduated from Yale in 1873, and had been teaching in that institution since 1876.

The fourth meeting of the Kansas Section was held at the University of Kansas on Saturday, March 17, 1917. An important program was carried out, a full report of which will appear in the June issue of the MONTHLY.

The Benjamin Peirce instructorships in mathematics at Harvard University have been filled by the reappointment of Dr. W. L. HART and the appointment of Dr. T. A. PIERCE. Mr. B. H. BROWN and Mr. J. L. WALSH have been appointed instructors in mathematics; and Dr. L. R. FORD, of the University of Edinburgh, has been appointed instructor in actuarial mathematics.

At the meeting of the Royal Society, February 22, one mathematical paper was presented, entitled "The ordinary convergence of restricted Fourier series," by Professor W. H. YOUNG.

Three papers of interest to teachers of mathematics appear in the March issue of *School Science and Mathematics* as follows: "Recent tendency in the teaching of elementary applied mathematics," by J. R. YOUNG; "Algebraic derivation of the law of cosines," by A. BABBITT; "A method of demonstrating and teaching the trigonometric functions," by S. C. MITCHELL.

On December 27–29, 1916, the Indian Mathematical Society held its first conference at Madras. An account of this meeting, the secretary's report and the presidential address are contained in the first number of Vol. IX of the *Journal of the Indian Mathematical Society*.

Professor A. GÉRARDIN, writing from Paris under date of February 24, 1917, reports the death of his two collaborators, Professors Barisien and Rudis, and the serious illness of Professor M. H. Brocard.

The MONTHLY still has on hand some copies of the reprints of Professor Huntington's paper on the "Logical skeleton of elementary dynamics," which may be had from the Secretary of the Association, Professor W. D. Cairns, 55 East Lorain St., Oberlin, Ohio, at ten cents each, postage prepaid.

The 191st regular meeting of the American Mathematical Society was held in New York on April 28, 1917. There were fourteen papers on the printed program, including one by Professor MAURICE FRÉCHET, of the University of Poitiers, and one by Professor LUIGI BIANCHI, of the University of Pisa.

Professor H. N. RUSSELL, Princeton University, has been appointed chairman of Section A (Mathematics and Astronomy) for the Pittsburgh meeting of the American Association for the Advancement of Science, to be held December 28, 1917, to January 2, 1918. Professor G. A. MILLER, University of Illinois, was elected from the Association at Large to represent Section A upon the Council for the same meeting.

Professor W. GILLESPIE, of the department of mathematics, at Princeton University, has gone with a party of students to the war zone for work under the direction of the Y. M. C. A. for the benefit of the English soldiers.

Professor R. C. ARCHIBALD, Brown University, writes that he would be glad to learn from any of the readers of the MONTHLY if a copy of a Russian translation of Steiner's little book, *Die geometrischen Konstructionen ausgeführt mittelst der geraden Linie und eines festen Kreises*, can be found in America. The book was translated into Russian by P. EROCHIN and R. HOLZBERG, and was edited by D. SINTZOR.

UNIVERSITY OF NEBRASKA. Summer session, June 11 to August 3. By Professor W. C. BRENKE: Calculus, five hours; Teachers' course, three hours; College algebra and plane trigonometry, five hours each.

UNIVERSITY OF ILLINOIS. The following mathematical courses are announced for the summer session of 1917: By Professor G. A. MILLER: Introduction to group theory; Theory of equations and determinants.—By Professor ARNOLD EMCH: Projective geometry; Constructive geometry.—By Dr. G. E. WAHLIN: Advanced calculus; Integral calculus.—Dr. E. W. CHITTENDEN: Differential calculus.—By Mr. H. D. FRARY: Analytical geometry.—By Mr. R. F. BORDEN: Plane trigonometry.—By Mr. C. H. RICHARDSON: College algebra.

UNIVERSITY OF WISCONSIN. Summer Session, June 25 to August 3.—By Professor C. S. SLICHTER: Mechanics, five hours; Hydrostatics, three hours; Algebra, five hours.—By Professor L. W. DOWLING: Theory of functions, three hours; Higher geometry, five hours; Analytic geometry, five hours.—By Assistant Professor ARNOLD DRESDEN: Calculus of variations, three hours; Definite

integrals, five hours; Calculus, five hours.—By Dr. T. M. SIMPSON: Differential equations, five hours; Commercial Algebra, five hours; Trigonometry, five hours. —By Assistant Professor W. W. HART: Determinants, three hours; Solid geometry, five hours; Teachers' course, five hours.

At the regular meeting of the London Mathematical Society held on February 1, the following papers were read: "The significance of a certain algebraic fraction in the theory of distributions" and "The number of ways of pairing off the members of two identical sets of different quantities," by Major MCMAHON; and "Curves of constant torsion," by W. H. SALMON. At the meeting held on March 1, papers were presented by A. E. JOLLIFFE on "Some properties of the quadrangle formed by the points of contact of the tangents to a nodal cubic," and by E. H. NEVILLE, on "Indicatrices of curvature."

The January, 1917, number of *The Rice Institute Pamphlet* has appeared containing the lectures delivered at the inauguration of Rice Institute by Professor EMIL BOREL, director of scientific studies at the Ecole Normale and professor of mathematics at the University of Paris, and by Senator VITO VOLTERRA, professor of mathematical physics and celestial mechanics at the University of Rome. Professor BOREL delivered lectures upon (1) "Aggregates of zero measure," and (2) "Monogenic uniform non-analytic functions"; Professor VOLTERRA's lectures were: (1) "The generalization of analytic functions" and (2) "On the theory of waves and Green's method."

It is remarkable that the French mathematicians can continue to issue a number of their periodicals. The May and June, 1916, numbers of the *Annales Scientifiques de L'École Normale Supérieure* have recently reached this country, containing lengthy memoirs. The January and February numbers of the *Bulletin des Sciences Mathématiques* are also at hand. Besides the reports and reviews these numbers of the *Bulletin* contain the following short papers: "Upon a theorem of Bertrand," by P. APPELL; and "A direct demonstration of the last theorem of Poincaré," by T. DANTZIG, of Indiana University.

The recently formed Mathematical Club at the University of North Carolina reports an enthusiastic membership numbering thirty. The plan of the programs is to meet once a month, having two papers presented, one upon a mathematical subject, and one upon some allied subject. The following subjects have been presented: "Mathematical requirements for electrical engineering students," "The differential coefficient," "The logic of mathematics," "Inscription of a 17-gon in a circle," "The teaching of mathematics in the high school" and "A note on linear equations."

In *Nature*, March 8, Professor JOSEPH LARMOR gives a short account of the life of the renowned French geometer, Professor GASTON DARBOUX, who died in Paris in February. DARBOUX was born at Nîmes, August 13, 1842; he re-

ceived his early training at the lycée of his native town, being engaged at the school from 6 A. M. to 8 P. M. Later he studied in the lycée at Montpelier, and in 1861 passed the examinations at the head of his class for admission to the Ecole Polytechnique and the Ecole Normale. Intending to engage in teaching, he chose the latter institution and began his studies with great inclination to the study of geometry, mastering the classic works of Monge, Gauss, Poncelet, Dupin, Lamé and Jacobi. In 1864 his memoir on orthogonal surfaces appeared in the *Comptes Rendus*, and two years later this same subject served as his dissertation for the doctorate in mathematical sciences at the Sorbonne. DARBOUX then began his career as a teacher of mathematics, first at the Collège de France, then at the Lycée Louis le Grand, and later as an assistant to Liouville at the Sorbonne. In 1880 he began his life work as professor of higher geometry at the Sorbonne, a position especially created for the renowned geometer Chasles, in 1846. During fourteen years following his graduation from the Sorbonne, DARBOUX published many original memoirs on partial differential equations, and theory of surfaces. In later years this early work was elaborated and published, 1887–1896, in the four classical volumes on infinitesimal geometry known as General Theory of Surfaces. DARBOUX was highly honored by being made a member of the Paris Academy of Sciences, 1884, and since 1900 its perpetual secretary. He was elected a foreign member of the Royal Society, 1900, and in 1916 received its high tribute by being awarded the Sylvester Medal.

The eighth regular meeting of the American Mathematical Society at Chicago was held on Friday and Saturday, April 6–7, 1917. In addition to sixteen papers on the program, there was held on Friday afternoon a symposium on "The Lebesgue Integral," conducted by Professor G. A. BLISS, of the University of Chicago, and Professor T. H. HILDEBRANDT, of the University of Michigan. Detailed synopses of these papers were printed with full references to the literature, and distributed in advance of the meeting. This symposium proved to be a most interesting and edifying part of the program. The dinner on Friday evening was, as usual, accompanied with informal talks on topics of interest to the mathematical fraternity, including particularly the new relation of the *Annals of Mathematics* to the Mathematical Association of America, and the contribution which the mathematicians of the United States might make to the government in the present war crisis. As the outcome of the discussion on the latter point, it was voted that the officers of the Society and of the Association present be requested to make recommendations to their respective Councils in the matter of organizing the resources of the mathematical constituency of the United States in the service of the government. Accordingly, the officers above mentioned joined in recommending that the Councils of the two organizations empower their officers to carry forward this program, in particular, to make a census of the members, if the government advises this, in order to list the various activities in which members are able, experienced, and willing to aid the government. The Council of the Association has since authorized such action as the officers deem wise by a practically unanimous vote.

At the Massachusetts Institute of Technology, a committee of the faculty has been appointed to consider ways of improving methods of instruction. Dr. C. R. MANN, who for the past two years has been preparing a report on engineering education under the auspices of the Carnegie Foundation for the Advancement of Teaching, has been called to the Institute to be chairman of the committee. In the *Educational Review*, January 17, Dr. MANN has published "A study of engineering education," reprinted from a Bulletin of the Society for the Promotion of Engineering Education. This is a very interesting report, tracing briefly the progress of education during colonial times, showing the increase of the demand for technical education, and the founding of the Rensselaer Polytechnic Institute in 1824, the first engineering school in the United States. The report then refers to the great industrial development of the period from 1825 to 1860, the Morrill Act of 1862 and the founding of the state colleges of mechanic arts in the years 1862–1875.

The report also shows some interesting studies on the capabilities of students in technical schools. Under the head of "What freshmen know and can do," it is shown that 90 per cent. of those tested could solve the simplest linear equation in algebra, while only one third of the freshmen could substitute and correctly reduce a simple, fractional expression containing x, a, b, when $x = (a + b) \div 2$. But in "What the schools do to freshmen," it appears that of 2,000 students who entered technical schools in 1911, only 732 graduated in 1915; the mortality in particular studies seems rather high, 52 per cent. passing in physics and a like number in mechanics, 45 per cent. in calculus, 43 per cent. in modern languages and English, and 34 per cent. in chemistry.

The large number of failures in the cases studied would amply justify the maintenance of such a committee as Dr. MANN is to head at the Massachusetts Institute of Technology to consider ways of improving methods of instruction. No doubt other institutions of higher education would profit in the effectiveness of their service to the people should similar investigations be instituted. Too often do instructors assume the attitude of examiner rather than that of real service to the students. The student has not learned how to work, and hence needs help. When 50 per cent., or 40 per cent., or 25 per cent., of a class in mathematics fail, the instructor should first be quite sure that the fault is wholly with the students.

Much interest is being manifested among colleges and universities in the maintenance of mathematical clubs for the especial benefit of undergraduate students. A large number of printed programs of these clubs has been received by the MONTHLY, and many inquiries have been made relative to material for such programs. The programs received by the editors are fashioned along similar lines, and include a range of subjects which should create much interest in the common knowledge of mathematics.

We append a collection of subjects suitable for club programs. (*A*) *Geom-*

etry: (1) Three historic problems; (2) "Flatland"; (3) Non-Euclidean geometry; (4) The fourth dimension; (5) The straight line; (6) Curve tracing; (7) The probability curve; (8) Imaginary loci; (9) Optical images; (10) The nine point circle; (11) Circles connected with a triangle; (12) Expressions for the area of a triangle; (13) Proofs of the Pythagorean theorem; (14) The planimeter; (15) Ancient geometry; (16) Modern geometry. (*B*) *Arithmetic and Algebra:* (17) Mathematical games; (18) The slide rule; (19) Magic squares; (20) Development of arithmetic and algebra; (21) Calculation of π and e; (22) Adding and multiplying machines; (23) Mathematical symbolism; (24) Numerical properties of color and sound; (25) Logarithms; (26) Mathematics of insurance and statistics; (27) Fermat's last theorem; (28) Theory of errors; (29) Mathematics of the calendar; (30) Properties of the numbers 9 and 37; (31) Number systems; (32) Remarkable numbers. (*C*) *Historical and Pedagogical:* (33) Mathematics of the Hindus; (34) Egyptian and Greek mathematics; (35) Mathematics of the Renaissance; (36) Japanese mathematics; (37) The development of trigonometry; (38) The German and French secondary schools; (39) Mathematics in the German universities; (40) Mathematics in American universities; (41) Vocational mathematics; (42) The development of the calculus; (43) Lives of great mathematicians (many subjects available); (44) Controversy between Leibnitz and Newton; (45) Definitions in mathematics; (46) "A Budget of Paradoxes," De Morgan; (47) Mathematical periodicals; (48) Zeno's arguments; (49) Opportunities for the study of graduate mathematics; (50) Why study mathematics?

Another series of topics is in preparation for publication and this is to contain references to the literature which may be helpful to clubs making use of the topics.

NOTES ON THE ASSOCIATION.

The membership of the Association grows with interesting speed. Since the Charter Membership list was completed on April 1, 1916, there have been added in September, 1916, 13 individual members and 8 institutional members; in December, 1916, 15 individual members and 16 institutional members; in February, 1917, 13 individual members and one institutional member; in April, 1917, 22 individual members and one institutional member. Thus the total additions to the Charter list are 63 individual and 26 institutional members. The names, hitherto unpublished, which were voted on by the Council in April, are as follows:

Kansas State Agricultural College to institutional membership.

Samuel Beatty, Lecturer, University of Toronto, Toronto, Can.

Sister Brigetta, Instructor in mathematics, College of St. Scholastica, Duluth, Minn.

H. N. Carleton, West Newbury, Mass.

Chancellor A. B. Chace, Brown University, Providence, R. I.

H. T. Davis, Instructor in mathematics, High School, Colorado Springs, Colo.

Dr. William S. Dennett, New York, N. Y.
Tomlinson Fort, Professor of mathematics, University of Alabama, University, Ala.
Mary E. Helwig, Instructor in mathematics, High School, Kansas City, Kansas.
Glenn James, Instructor in mathematics, Purdue University, West LaFayette, Ind.
H. P. Kean, Professor of mathematics, McKendree College, Lebanon, Ill.
C. H. Lehmann, Student, Cooper Union, New York, N. Y.
J. S. Mikesh, Director, Junior College, Hibbing, Minn.
O. J. Ramler, Instructor in mathematics, Catholic University of America, Washington, D. C.
W. H. Rittenhouse, M.E., Philadelphia, Pa.
S. T. Sanders, Assistant Professor of mathematics, Louisiana State University, Baton Rouge, La.
W. G. Simon, Graduate student, University of Chicago, Chicago, Ill.
C. E. Smith, Professor of mathematics and science, Northland College, Ashland, Wis.
G. T. Street, Instructor in mathematics, Denison University, Granville, Ohio.
H. R. Tolley, Scientific assistant, Office of Farm Management, U. S. Department of Agriculture, Washington, D. C.
T. E. Trott, Professor of mathematics, Mt. Union College, Alliance, Ohio.
J. K. Whittemore, Instructor in mathematics, Sheffield Scientific School, Yale University, New Haven, Conn.
E. H. Worthington, Instructor in mathematics, University of Pennsylvania, Philadelphia, Pa.

The Committee on Libraries, of which Professor W. B. Ford is chairman, has made good progress and will be able to report very soon, possibly in time for the June issue, certainly for the September issue.

The Committee on Fortschritte and Revue Semestrielle, of which Professor E. V. Huntington is chairman, has been able to make very little progress. The only information the committee has been able to gather comes from a London bookdealer, who stated on April 12, 1917, that he had been advised that numbers 1 and 2 of volume 24 of the Revue Semestrielle had appeared. Number 1 of volume 25 has not been issued so far as is now known.

The Program Committee and the Committee on Arrangements for the Summer meeting in Cleveland on September 7 and 8 will make preliminary reports for publication in the June MONTHLY, which will be mailed to members during the last week in June. The June issue will be the last to appear previous to the meeting. The final announcements will be mailed about the middle of August under second-class postage to the addresses of members on the MONTHLY mailing list, unless the Secretary is advised of changes of addresses prior to that time.

The Kentucky Section of the Mathematical Association of America has been organized. The teachers of higher mathematics in Kentucky have maintained a local state organization for nine years; this organization now becomes a part of the larger Association, having been an early applicant for admission as a section. Professor E. L. REES, of the University of Kentucky, is secretary of the section. The spring meeting of the section was held on May 11, 1917, at Berea, Ky.

The Rocky Mountain Section of the Association has been organized and will include the states of Colorado and Wyoming. The first meeting was held on April 7, 1917, and will be reported in the June issue. There are now nine sections of the Association, namely, in Kansas, Ohio, Missouri, Iowa, Indiana, Kentucky, Minnesota, Maryland, Virginia, District of Columbia, and Colorado-Wyoming. Many of the sections hold meetings twice a year. Two spring meetings of sections are reported in this issue, and four more will be reported in the June issue.

The editors of the *Annals of Mathematics* report a most gratifying response to the circular sent out by the Association in March announcing the enlargement of the *Annals* for 1917–1918, and asking for subscriptions on the new basis of cooperation between the *Annals* and the Association. Over three hundred and thirty new subscriptions had been received when this was written, aside from sixty or seventy requests for samples copies of the issue containing Professor Dickson's article on the Fermat Theorem. The new volume of the *Annals* on the enlarged basis begins in October, 1917, and subscriptions from members of the Association at the one-half rate ($1.50) may be sent at any time to the Editors at Princeton, New Jersey. Payment may be deferred, if desired, until October first. This opportunity is open to new members of the Association whenever they may join.

The Secretary and Managing Editor have just made up a shipment of several boxes of books and journals to the library of the Association, which is at present to be housed in a private alcove of the library of Oberlin College, where it will be under the supervision of the Secretary and will be catalogued and cared for by the library staff of Oberlin College. The journals comprise the exchange list of the MONTHLY, which is already well extended and is slowly growing, together with certain sets of journals extending back varying periods which have been contributed by the Managing Editor. These include the *Bulletin of the American Mathematical Society*, *School Science and Mathematics*, *The Mathematics Teacher*, and *Science*. It is hoped that these and other partial sets may be completed. A list of copies or volumes wanted will be published as soon as an inventory can be made of those in hand. The complete set of the MONTHLY contributed jointly by Professors E. H. MOORE and H. E. SLAUGHT has now been bound and deposited in the library.

The number of books is not yet extensive, but it is hoped that all publishers

in this country who print college and university mathematical books will be interested to contribute copies to the library. In order to insure proper credit to all such donors the Secretary has prepared a suitable printed label to be attached to the inside front cover of all books contributed. These will be supplied to publishers, to authors, or to any members or friends of the Association who may wish to become donors. All such contributions may be sent to The Mathematical Association of America in care of the Oberlin College Library, Oberlin, Ohio.

In particular, any persons who may have back numbers or volumes of the MONTHLY, especially back of 1910, are requested to send an inventory of the same to the Secretary. He is continually asked to supply missing copies for those who are making up sets, and he will pay cash for these whenever they can be resold.

SPECIAL ANNOUNCEMENT.

The MATHEMATICAL ASSOCIATION of AMERICA invites all members who have constructive ideas upon the topic: "The treatment of the applications in college courses in mathematics," to submit brief papers on that subject before June 15. All papers will be passed upon by the program committee and a suitable selection will be made for presentation at the summer meeting. This is a subject that should arouse much interest. Address all communications to the Chairman of the Committee, Professor C. S. Slichter, University of Wisconsin, Madison, Wis.

The following from The New Era Printing Company explains the delay of the April issue:

"The delay was caused by the slow freights. We had exhausted the supply of cover paper, and while an order had been placed in ample time and we received the bill of lading on the 9th of April, the paper did not reach us until the last of the month. We regret the delay which was unavoidable."

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ALGEBRAICAL DEVELOPMENTS AMONG THE EGYPTIANS AND BABYLONIANS.

By LOUIS C. KARPINSKI, University of Michigan.

INTRODUCTION.

The purpose of this discussion of algebraical developments among the Egyptians and Babylonians is to show that much of the material of our elementary algebra was long ago anticipated, to some extent, in the Orient. Similar anticipations of algebraical reasoning are indicated in the material, such as we have, which shows the progress of mathematics in ancient India and China and Greece.

In Egypt and India and probably in Babylon, too, the numerical phases of mathematical problems received the greater attention; in Greece the geometrical phases were primarily studied. However, by these diverse routes many of the same problems were studied. The first degree equation in one unknown quantity appeared in analytical garb in Egypt, but the same problem is solved geometrically in Greece; simultaneous quadratics (leading to pure quadratics) appear in old Egyptian papyri, solved numerically, while similar problems are given geometrical solution by the ancient Greeks. Numerical application of the Pythagorean proposition appears before the time of Pythagoras in Egypt and probably in India, while the triumph of a logical, geometrical proof is reserved for the Hellenic people as a part of their most noteworthy contribution to the development of human intelligence. The study of arithmetical and geometrical progressions took a somewhat analogous course, appearing on the Ganges, on the Nile, on the Tigris and Euphrates, and, in most complete development, by the Aegean Sea. The conclusion which is suggested by the concordant development of mathematics in ancient times among diverse civilizations, as well as by the dominant position